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Geometry
I.C!

ELEMENTS

OF THE

GEOMETRY

OF

PLANES AND SOLIDS.

WITH FOUR PLATES.

BY

F. R. HASSLER, F. A. P. S.

RICHMOND.

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1828.

Eastern District of Virginia, to wit:

Be it remembered, That on the twelfth day of July, in the fifty-third year of the Independence of the United States of America, FERDINAND R. HASSLER, of the said District, hath deposited in this Office the title of a book, the right whereof he claims as author, in the words following, to wit:

L. S.

"Elements of the Geometry of Planes and Solids. With four Plates. By F. R. Hassler, F. A. P. S."

In conformity to the Act of the Congress of the United States, entitled "An act for the encouragement of learning, by securing the copies of maps, charts and books, to the authors and proprietors of such copies, during the times therein mentioned."

R'D. JEFFRIES, *Clerk of the Eastern District of Virginia.*

INTRODUCTION.

GEOMETRY, is that branch of *Mathematics* which treats of the *magnitude* and *relation of figures*, in the most general acceptation of the word. Its immediate subject is, therefore, *Dimension*, which it *compasses* in any *form* and *magnitude* that may be required; in order to represent exactly, or approximate as near as desired, any *extension*, *surface* or *solid*, presented in nature, or imagined with the view of application to nature, and the relations of which it determines. Elementary Geometry treats of these *relations* only *as limited:* higher Geometry determines them *without reference* to limitation.

The first elements of Geometry take their rise in *principles* so *strong* and so *simple* as to be *undeniable* and *self evident*, before any scientific form is given to them. It, therefore, begins by a simple *statement* of these, under the denomination of *Axioms;* or if not actually stated, they are tacitly supposed as admitted.

Then, under the title of *Postulates*, it requests the admission, (*intellectually*,) that a *straight line* may be drawn, and a *circle* described, both of which it supposes exact, without entering on the question how near this accuracy may be attainable in the mechanical practice.

Upon these few and simple data the science is built, by the synthetical process of reasoning, to a full system, presenting all the necessary means to apply its results to nature; either by the construction of them, according to its own principles, (which is practical Geometry,) or by delivering its results over to the calculation, as *magnitudes*, the relations of which it has determined.

While, therefore, the Elements of *Arithmetic* present us with the principles of *exact reasoning*, in the *analytical* form,

the Elements of *Geometry* must present us with the same, in the *synthetical* form. Together, therefore, they present a full system of *exact logic*, and always *fully exact results*, at which all reasonings upon any other subject in the material, intellectual and moral world must aim, and the degree of accuracy of the results in these cases is decided by the proximity of the reasoning to either one of the two courses thus pointed out by these two elementary branches of Mathematics.

Every science must necessarily determine its language, that is, *define* the *exact value* of those words that are peculiar to it. Geometry has evidently to do the same; it therefore defines, in that manner in which it makes use of them, the denomination of certain lines and the few simple figures of which it makes use, to deduce all its results upon any figure presented to its inquiry, and in which it decomposes these, in its application to practice.

To build up the system of the science, by thus *synthetically* proceeding from the known to the unknown, it is *absolutely necessary*, that no step be taken the exact truth of which has not been previously demonstrated, or where it has not been shewn: how the requisites are obtained with absolute exactitude, satisfactory to the intellect. Thence the *Propositions* must all be individual as to the principle proved, or the operation to be performed, under the conditions stated by this proposition.

The *order* of these Propositions is unavoidably determined; the *Theorems* and *Problems* must follow upon each other exactly in that *alternating order*, in which the means are obtained for their *demonstration* or *solution*, and in which they become necessary for the further progress: any inversion of this order would be a direct offence against the *principles* of the science, because it breaks the chain of exact and close connection, which constitutes the strength of this science, and would deprive the superstructure of its foundation. The only liberty left in this respect, is in the collateral consequences which may follow from a proposition, which, in case of peculiar aims, may be chosen with reference to these. But, in treating the first

Elements of the *Science in general*, the course and its order are fully determined, and limited by the *peremptory law* of every elementary work to lead by the shortest road; and to present a *full system* of the *fundamental elements of the science*, in the greatest *simplicity* compatible with full *exactness*, and without any luxury of knowledge, by the introduction of accessories that might appear pretty, and which will occur by themselves to him who possesses the system of the science.

Without such a course, the present extensive field of Mathematics would present itself to a scholar under such a magnitude and multiplicity of interwoven ramifications, that it would appear impossible to him to acquire the science and to abstract its principles from them.

These principles shall guide me in the following Elementary Geometry. I propose to myself to lead the scholar by the simplest, and clearest, steps, in the *form* which this science has taken more than two thousand years ago, (generally known under the name of *euclidean*, the correctness of which has so much distinguished and elevated this *science*,) to that point, where the data being obtained, which enable to treat the geometric figures and lines under the general form of *quantities*, it is proper, in the present state of Mathematics, to treat the further subjects in the analytical form; that is, by application of general Arithmetic to the principles of their equality and proportionality, demonstrated before; and therefrom to deduce their absolute determination, under the mere supposition of an Unit of the kind required, as line, surface, or solid.

When the proportionality of figures is treated, it is necessary to suppose the knowledge of Arithmetic: I take, therefore, the liberty to suppose my reader acquainted with my Elements of Arithmetic, and particularly, with the principles of *ratio* and *proportion*, as there explained.

It will be proper here to give a clear idea of the distinct nature of the different propositions made use of in Geometry, their parts, and the principles of their modes of demonstration, or proof.

A PROPOSITION: is generally, a subject proposed, either by way of an *assertion* of a principle, that it is the object of the proposition to *prove true;* or by the *execution* of an *operation of Geometry*, by means of constructions, the *correctness* of which is *proved.* The first is called a *Theorem*, the second a *Problem.*

A THEOREM must necessarily contain the following parts:

1. The *Enunciation*, in general terms, of the *truth proposed*, and intended to be *proved.*

2. The *application* to one or more figures, presenting the *positions* enounced, or the case supposed in the Theorem.

3. The *Construction* that may be needed for the demonstration of the truth, from the principles of previous propositions, or truths known in general.

4. The *Deduction* of the consequences of the construction.

5. The *Proof* which these Deductions afford for the truth proposed in the Theorem.

6. The *Conclusion*, which states the consequences of the proof upon the *confirmation* of *the truth* proposed.

The parts under 3 to 6, are generally comprehended together under the name of *Demonstration.*

A PROBLEM, must by its nature consist of the following parts:

1. The general *Enunciation* of the *Operation* or *construction* which is requested to be done, and the nature of the *means* or *data* given for its execution.

2. The *Application* to a *figure* or case, and certain given *Data.*

3. The *Solution*, or the actual construction of the Problem, or the performing of the Operation with the given Data.

4. The *Construction* that may be needed to shew the consequences of the Operations performed.

5. The *Deduction* of the consequences of the *Solution* and the *construction* upon the correctness of the Operation performed, as grounded upon previously demonstrated truths.

6. The *Conclusion:* that the Operation performed has been *executed*, and is *exact*, according to principles *previously* demonstrated.

The parts under 4 to 6, are generally comprehended together under the name of *Proof*.

If from a Theorem or a Problem some more consequences follow: that become evident without needing further *special proof*, they are added to them under the title of *Corollaries*.

Under the name of *Scholium* may be added such conclusions from a *Proposition* as become evident by its *comparison* with another, previously demonstrated, without needing *special Demonstration*.

When for one or more propositions *to be* treated, it is necessary to have recourse to *consequences* of such as have been proved elsewhere, or are considered as previously known, without being introduced in the subject to be treated; the truth so made use of is prefixed under the name of a *Lemma*.

The methods of proving in Mathematics are like all other proofs whatever, according to the circumstances of the case, either *direct*, or by (what is called) *anti-thesis*. The *direct proof* proceeds forwards, from *previously averred truth*, step by step to the demonstration of the assertion which it shall support; in the same order in which the science furnishes the means for the proposed demonstration. The *anti-thesis* proves: that any other position, than that advanced, leads, according to principles of the science previously proved, to an *absurdity*, by contradicting either *itself*, or any previously *demonstrated truth*.

In the *Theorem* both these methods of proof are admissible, with equal strength of the conclusions; because the assertion must hold equally, when its *contradiction* is proved *impossible*, and when its *assertion* is proved *true*. In the *Problem*, only the direct proof can be admitted, because the thing to be done must be executable, which is not the case in the anti-thesis.

In the *anti-thesis* it is necessary that *all* cases of *contradiction* that are possible be exhausted, and reduced to absur-

dity, thence this method is often also called "method of exhaustion," in pure Mathematics, which grounds all its conclusions upon *equality;* this is reduced to the two simple supposition of either *greater* or *less*, which being proved to lead to a contradiction of themselves are inadmissible, and therefore the equality alone admissible.

It may be proper here to state those *axioms* which are general to all Mathematics, as they relate to quantity in general; and which, for that reason, have not been inserted in the body of the work itself, where I found best to insert only those two, which are peculiar to Geometry.

The others form by their nature *Lemata* for the Geometry: they may be expressed thus:

1. The *whole* is equal to *all* its parts taken together, thence also greater than any of its parts.

2. Things or *quantities*, which are equal to one and *the same*, other thing, or *quantity*, are equal to one another.

3. *Equals added* to equals give equal quantities.

4. The *Differences* of equal quantities are equal.

5. Things and quantities which are equal parts of the same, or equal things or quantities, are equal.

6. *Quantities* which contain an equal number of the *same*, or *equal, quantities* are equal to one another.

I considered myself authorised to omit several definitions of distinctions of figures or solids, as useless, or even improper; the first, because they can be entirely spared, and are, actually, not separately treated of in Geometry; the other, because the reader, being at the beginning not supposed to know their nature, can take no clear idea from the definition, which he acquires only from the subsequent propositions, where I have introduced them, when the origin or nature of the figure is shewn; as for instance: trapezium, oblong rectangle, acute, or obtuse, angled triangle (before the scholar knows that a triangle cannot have more than *one* right-or obtuse-angle.) The definition of parallel lines cannot be objected to: as it is a mere *nominal definition of a certain supposition*, the consequences of which are afterwards proved, exactly as that of

three lines cutting each other forming a triangle. Thence, also, the *origin* of the parallelogram is shewn, and it is not defined. To say that all right angles are equal, appeared to me to say no more than equal things are equal.

In plane Geometry we proceed from the simple to the compound, and in all practical applications of Mathematics we treat every compound quantity or figure by its component parts; by following these steps also in solid Geometry exactly parallel to their similar ones in plane Geometry, I found the path ready made before me, though different from the one usually adopted. A scholar cannot abstract immediately in a number of planes, which besides is not synthetical, as all the rest of Geometry is; but seeing the solids in some measure created under his eyes, he meets with no difficulty in comprehending the solids thus resulting, as he has seen the similar results in plane Geometry.

The sub-divisions of the chapters were made with the view to shew to the scholar, distinctly, the nature of the subjects of the enquiry, which therefore must guide the mode of proceeding in each part; and I aimed at approaching to the practical usefulness and application, as near as possible without numerical examples.

To enter into details, or the reasons which decided in the one or the other particular parts, would make a dissertation of no real interest to the man of science; and the scholar is of course a stranger to the whole of the polemic part of the science; thence it would be improper here. I commit, therefore, this work simply to the reflection of the clear head, and the reflecting mind, wishing that it may answer the aim of utility for which it is intended, to render easy the elements of that science, the unerring principles of which should guide man in his actions, as it guides the Universe in its motions, and secures its *stability*.

F. R. HASSLER.

Richmond, Virginia, June, 1828.

PART I.

PLANE GEOMETRY.

CHAPTER I.

PRELIMINARIES.

DEFINITIONS.

1. A *point*, is that which has no magnitude nor parts.

2. A *line*, is length without breadth or thickness.

3. A *straight line*, is that in which any two points have the same direction with any other point of the line however taken.

4. A *surface*, is that which has length and breadth without thickness.

5. A *plane surface*, is that in which any two points being taken, the straight line between them lies wholly in that surface.

6. A *rectilineal angle*, is the inclination of two straight lines that cut each other. (Fig. 1.)

7. When two lines cut one another so as to make the angles on the opposite sides of the same line equals, these lines are said to be at *right angles*, or *perpendicular* to one another. In figure 1, *BAD*, and *CAF*, are at right angles to one another when the angles *CAB*, and *CAD*, are equal, or the angles *CAB*, and *BAF*.

8. An *obtuse angle*, is that which is *greater* than a right angle. In figure 1, the angle EAB, is an obtuse angle.

9. An *acute angle*, is that which is *less* than a right angle. In figure 1, the angle EAD, is an acute angle.

10 *Figure*, is any space terminated. (So figure 2 to 8 are figures in that sense, as they inclose or terminate a space.)

11. A *circle*, is a *plane figure* terminated by one single line which is every where equally distant from a certain point within it, which is called the centre. Figure 2, is a circle of which C, is the centre.

12. The *circumference* of the circle is the curve line which terminated the figure. The curved line $DAEFB$, in figure 2.

13. The *radius* of the circle is the straight line between the centre and the circumference of the circle, CD, CA, CB, in figure 2, are radii, and these are all equal.

The circle is therefore described by the revolution of this line, or the radius, around one of its end points, which remains fixed as centre, while the other end point describes the circumference of the circle.

14. The *diameter* of the circle is the line which, passing through the centre of the circle is terminated on both sides by the circumference, AB, in figure 2.

15. A *chord*, is any straight line drawn in the circle and terminated on both sides by the circumference. Like EF, in figure 2.

16. A *sector*, is a part of a circle contained by two radii and a part of the circumference of a circle, as ACD, in fig. 2.

17. A *segment* is a part of a circle contained by a chord and a part of the circumference; as EGF, in figure 2.

18 *Rectilineal figures* are such as are terminated by straight lines; as are the figures 3 to 8 inclusive.

19 A *plane triangle* is a figure terminated by three straight lines, (the figures 3 to 6 are triangles) three straight lines cutting one another form a plane triangle, and they are the smallest number that can include a space, that is form a rectilineal figure.

20. A *plane quadrilateral* figure is that which is terminated by four straight lines, that lie in the same plane. All rectilineal figures of more sides take their denomination from the number of their sides, as figures 7 and 8.

21 An *isosceles triangle*, is that which has two of its sides equal; as figure 4.

22. An *equilateral triangle*, is that which has all its three sides equal; as figure 3.

23. A *right angled triangle*, is that which has one of its angles a right angle; as figure 5.

24. A *square* is a figure of four sides having all its sides equal and its angles right angles; as figure 7.

25. *Parallel straight lines* are such as lie in one plane; and being protracted in both directions to any extent, will never cut one another; as figure 9.

POSTULATES.

It must be granted that:

1. A *straight line* can be drawn between any *two* given points.

2. A *straight line* can be protracted to any desired extent.

3. A *circle* can be described *with any* line as *Radius*.

AXIOM.

1. Magnitudes which exactly coincide, or cover one another, are equal.

2. Two straight lines cannot include a space.

CHAPTER II.

Of Straight Lines; first Elements of Rectilineal Figures and equality of Triangles.

§ 1. PROBLEM. To describe an equilateral triangle with a given straight line.

APPLICATION. *Let* AB, *be the given straight line with which it is required to describe an equilateral triangle.* (*Fig.* 10.)

Solution. From the point *A*, as centre, with the line *AB*, describe the circle *BCD;* and from the point *B*, as centre, with the same line *AB*, describe the circle *ACE;* from *C*, a point of intersection of the circumferences of these two circles draw the straight lines *CA*, and *CB*, to the two extremities of the line *AB*, the triangle *ABC*, will be the equilateral triangle required.

Proof. The circumferences of the two circles *BCD*, and *ACE*, must cut one another, because the centre of the one lies in the circumference of the other, so that the circle must lie partly within and partly without one another; the point *C*, where these circumferences cut one another being common to both, the straight line *AC*, is equal to *AB*, as radii of the circle *BCD*, and the straight line *BC*, equal to the *AB*, as radii of the circle *ACE;* therefore, the two straight lines *AC*, and *BC*, are both equal to *AB*, therefore they are equal to one another; that is, all the three sides of the triangle *ABC*, are equal, or, it is equilateral, as was to be proved.

§ 2. PROBLEM. From a given point to draw a straight line equal to a given straight line.

APPLICATION. *Let* AB, *be the given straight line; C, the given point, from which it is required to draw a straight line equal to the given straight line* AB. (*Fig.* 11.)

Solution. Join *AC*, and with that line describe the equilateral triangle *ACD*, protract *DA*, and *DC*, (indefinitely,) from *A*, as centre, and with *AB*, as radius describe the circle *BEG*, which will cut the protracted *DA*, because the centre *A*, lies in it; let this point of intersection be *E;* from the centre *D*, with *DE*, as radius describe the circle *EFH*, which will cut the protracted *DC*, because the centre *D*, lies in it, let this point of intersection be *F;* the line *CF*, will be the straight line required, drawn from the point *C*, and equal to *AB*.

Proof. The straight lines *AE*, and *AB*, are equal as radii of the circle *BEG;* the sides *DC*, and *DA*, are equal as sides of the equilateral triangle *ADC;* the lines *DAE*, and *DCF*, are straight lines being the protractions of the straight lines *DA*, and *DC;* the lines *DE*, and *DF*, are equal as radii of the same circle *EFH;* taking from each of the equal straight lines *DF*, and *DE*, the equal lines or sides of the triangle *DC*, and *DA*, respectively; the remainder *CF*, will be equal to the remainder *AE;* the *AE*, being equal to *AB*, the *CF*, is also equal to *AB;* therefore from the point *C*, has been drawn a line *CF*, equal to *AB*, as was required.

Corol. As from the centre *C*, with the straight line *CF*, a circle can be drawn, the line equal to *CF*, or *AB*, may be laid in any direction that may be desired.

§ 3. PROBLEM. **From the greater of two given straight lines to cut off one equal to the smaller.**

APPLICATION. *Let* AB, *be a given straight line larger than* CD, *it is required to cut off from* AB, *a line equal to* CD. (*Fig.* 12.)

Solution. From the point *A*, lay off the straight line *AE*, in any direction and equal to the given line *CD*, from the centre *A*, with *AE*, as radius describe a circle or section of a circle *EFG*, which will cut the *AB*, in *F*, the line *AF*, will be the line required cut off from *AB*, and equal to *CD*.

Proof. The straight line *AE*, being made equal to *CD;* and the *AF*, being equal to *AE*, as radius of the same circle, the *AF*, is equal to *CD*, and because *CD*, is smaller than *AB*, the circle described with *AE*, or *AF*, must cut it, therefore the *AE*, is cut off from the *AB*, equal to *CD*, as was required.

§ 4. THEOREM. **If two triangles have two sides in the one equal to two sides in the other, each to each, and the angle between these two sides in the one triangle equal to the angle between the equal sides in the other triangle, the third side of the one will also be equal to the third side of the other; the angles opposite to the equal sides in both triangles will be equal; and the triangles will be equal in all respects.**

APPLICATION. *Let* ABC, *and* DEF, *be two triangles* (*figure* 13,) *having the side* AB, *in the one, equal to the side* DE, *in the other; the* AC, *in the first, equal to the* DF, *in the second, and the angle at* A, *equal to the angle at* D; *then will also be: the side* BC, *equal to the side* EF; *the angle at* B, *equal to the angle at* E, *the angle at* C, *equal to the angle at* F; *and the triangles* ABC, *and* DEF, *will be equal in all respects.*

Demonstration. Lay the triangle *ABC*, upon the triangle *DEF*, so that the point *A*, fall upon the point *D*, the line *AB*, upon the line *DE;* then will the line *AC*, fall upon *DF*, because the angle *A*, is equal to the angle *D;* the point *B*, will fall upon the point *E*, because *AB*, is equal to *DE;* the point *C*, will fall upon the point *F*, because the line *AC*, is equal to *DF;* therefore also the lines *CB*, and *EF*, which join these two points will coincide, as they lie between the same points, and two straight lines cannot include a space. Also the angle *B*, will coincide with the angle *E*, and the angle *C*, with the angle *F;* therefore, the angles opposite to the equal sides will be respectively equal in the two triangles; therefore all the

parts of the two triangles will coincide and the whole triangles will be equal to one another in all respects, as was to be demonstrated.

Corol. This proposition shews that two sides and the included angle determine a triangle, for with these parts a triangle can be made equal to one having these data.

§ 5. THEOREM. In an isosceles triangle the angles opposite to the two equal sides are equal; and if these sides are protracted, the angles which these protractions make with the third side, on the outside of the triangle are also equal.

APPLICATION. *Let* ABC, (*fig.* 14,) *be an isosceles triangle, in which* AB, *is equal to* AC; *then will the angle* ABC, *be equal to the angle* ACB; *and when* AB, *and* AC, *are protracted on the other side of* BC, *the angle* CBF, *will be equal to the angle* BCG, *outside of the triangle.*

Demonstration. In the protracted *AF*, take any line *AD*, greater than *AB*, and upon the protracted *AC*, make *AE*, equal to *AD;* join *BE* and *CD*. In the two triangles *ABE*, and *ACD*, the two lines *AB*, in the one, and *AC*, in the other, are equal, being the equal sides of the isosceles triangle *ABC;* the line *AE*, equal to *AD*, by construction; and the angle at *A*, is common to these two triangles; therefore the third side *BE*, is equal to the third side *CD*, and the angles of these triangles that are opposite to equal sides, are equal; that is, the angle *BEA*, equal to the *CDA;* and *EBA*, equal to *DCA*. In the two triangles *BCD*, and *CBE*, the sides *BD*, and *CE*, are equal, because they are the differences of equal sides, *AD*, and *AB*, and of *AE*, and *AC;* the angle at *D*, has been proved equal to the angle at *E*, therefore these two triangles are equal in all respects, and the angles *CBD*, and *BCE*, are equal; these are the angles of the protracted equal sides, with the third side outside of the triangle. Also the angles *EBC*, and *DCB*, are equal; therefore, if from each

of the angles ABE, and ACD, which have above been proved to be equal, these equal angles are taken respectively: the remainders ABC, and ACB, will be equal; and these are the angles of the triangle, which are opposite to the equal sides. Therefore, in an isosceles triangle, the angles which the third side makes with the equal sides, are equal, and if the equal sides are protracted, also the angles on the other side of this third side, are equal.

Corol. Every equilateral triangle is also equiangular, because the above demonstration applies to any two sides indiscriminately.

§ 6. THEOREM. **If a triangle has two of its angles equal, the sides opposite to these equal angles will be equal.**

APPLICATION. *Let* ABC, *be a triangle in which the angle* ABC, *be equal to the angle* ACB, *then will also the side* AC, *be equal to the side* AB, (*fig.* 15.)

Demonstration. If they are not equal, the one of them must be greater than the other; suppose AB, the greater, let BD, be cut off from BA, and equal to AC; join DC; then in the two triangles ACB, and BDC, there will be the side AC, equal to BD; the side BC, common to both triangles, and the angle ACB, equal to the angle DBC; therefore, the two triangles ABC, and DBC, are equal, and the angles ACB, and DCB, being both equal to the ABC, they are equal to one another; that is, the smaller equal to the greater, which is impossible; therefore, the sides AB, and AC, are not unequal; that is, if two angles in a triangle are equal, the sides opposite to these angles are also equal.

Corol. Every equiangular triangle is also equilateral, because the above reasoning applies to all the three angles equally.

§ 7. THEOREM. **On the same side of a straight line there cannot be two different triangles, which have the two sides ending in the same points of that straight line equal to one another.**

APPLICATION. *If possible, let* BCA, *and* BCD, (*fig.* 15, 16 *and* 17,) *be two such triangles, having the side* BC, *common, and the points* A, *and* D, *not coinciding, though their sides* AB, *and* AC, *be respectively equal to* DB, *and* DC, *ending in the same points* B, *and* C, *the point* D *must fall either in one of the lines, or within the triangle, or without the triangle.*

Demonstration. I. If (fig. 15,) the point *D*, falls in the *BA*, then *BD*, would be equal to *BA*, the part equal to the whole, which is impossible.

II. If the point *D*, falls within the triangle *ABC*, (fig. 16,) join *AD*, and protract the equal lines *BD*, and *BA*, to *F*, and *E;* the triangle *BAD*, is isosceles, and the angles *EAD*, and *ADF*, are equal, as outside angles of this isosceles triangle; the angle *EAD*, is greater than *CAD;* therefore, also, *ADF*, is greater than *CAD;* therefore, much more *CDA*, is greater than *CAD*, but *CAD*, and *CDA*, are equal, as angles opposite to the equal sides in the isosceles triangle *CAD;* these two angles are, therefore, at the same time greater the one than the other, and equal the one to the other, which is impossible; therefore, the point *D*, does not fall within the triangle *ABC*.

III. If the point *D*, falls without the triangle *ABC*, (fig. 17,) join *AD;* the sides *AB*, and *DB*, are equal by the supposition; therefore, the angle *BAD*, is equal to *ADB;* the sides *AC*, and *DC*, being also equal by the supposition, the angles *CDA*, and *CAD*, are also equal; but the angle *BAD*, is greater than *CAD;* therefore, also, *BDA*, is greater than *CAD*, so much more *ADC*, must be greater than *CAD;* but these angles are also equal; therefore they are at the same time equal and greater the one than the other, which is impossible; therefore the point *D*, does also not fall outside of the triangle *ABC*, or the points *D*, and *A*, must coincide, and there can be but one triangle on one side of *BC*, the sides of which ending in the same points, *B*, and *C*, are equal; or the triangles must be identical.

§ 8. THEOREM. **When two triangles have the three sides of the one respectively equal to the three sides of the other, each to each, the angles in the one will be equal to the angles in the other, each to each; those, namely, which are opposite to the equal sides, and the triangles will be equal in all respects.**

APPLICATION. *Let* ABC, *and* DEF, (*fig.* 13,) *be two triangles: having the side* AB, *equal to* DE, *the* AC, *equal to* DF, *and the* BC, *equal to* FE; *then will the angle* B, *be equal to the angle* E, *the* A, *equal to* D, *and the* C, *equal to* F; *and the triangle* ABC, *will coincide with the triangle* DEF.

Demonstration. Place the triangle *ABC*, upon the triangle *DEF*, so that the point *A*, fall upon the point *D*, and the line *AB*, upon the line *DE*, then will the point *B*, fall upon the point *E*, because the *AB*, is equal to *DE*, the point *C*, will fall upon the point *F*, because *AC*, and *BC*, are equal to *DF*, and *EF;* the triangle *ABC*, will coincide with the triangle *DEF*. For, if the point *C*, did not fall in the point *F*, it must fall in any other point within or without the triangle *DEF*, and there would be on the same side of *DE*, two different triangles having two sides ending in the same points *D*, and *E*, of the third sides equal, which has been proved impossible by the preceding Theorem; therefore, the point *C*, must coincide with the point *F*, the side *AC*, with *DF*, and *BC*, with *EF;* the angle at *A*, will coincide with the angle at *D;* the *B*, with the *E*, and the *C*, with the *F;* and these are the angles opposite to the equal sides in the two triangles; therefore, to the equal sides the equal angles are opposite, and the triangles are equal in all respects.

Corol. When the three sides of a triangle are given, the triangle is determined.

§ 9. PROBLEM. **To divide a given rectilineal angle into two equal parts (or angles.)**

APPLICATION. *Let* ACB, (*fig.* 18,) *be a given angle; it is required to divide it into two equal parts, that is to* bisect it.

Solution. In *CB*, take any point, as *D*, and from *CA*, cut off the *CE*, equal to *CD;* join *DE*, and upon it describe the equilateral triangle *DEF*, draw *CF*, this line will bisect the given angle; that is *BCF*, will be equal to *ACF*.

Proof. In the two triangles *CDF*, and *CEF*, the line *CD*, is equal to *CE*, the *DF*, equal to *EF*, by construction, and the *CF*, common to both triangles; these triangles have, therefore, their three sides equal each to each; therefore they are equal; and the angles opposite to the equal sides are equal; that is also the angle *DCF*, is equal to the *ECF*, these two forming together the given angle *BCA*, this angle is bisected by the line *CF*, as was required.

§ 10. PROBLEM. To divide a given terminated straight line into two equal parts, (that is to bisect it.)

APPLICATION. *Let* AB, (*fig.* 19,) *be a given terminated straight line it is required to bisect it.*

Solution. Upon the line *AB*, describe the equilateral triangle *ABC*, bisect the angle *ACB*, by the straight line *CF*, the intersection of *CF*, with *AB*, in the point *D*, will bisect the line *AB*, or make *AD=BD*.

Proof. In the two triangles *ACD*, and *BCD*, the line *CA*, is equal to *CB*, the *CD*, is common to both triangles, and the angle included between these equal lines in the one triangle is equal to the angle included by the equal lines in the other; therefore, also, the third side *AD*, in the one triangle is equal to the third side *BD*, in the other, and the *AB*, is bisected in *D*, as was required.

§ 11. PROBLEM. From a given point in a given straight line to draw a perpendicular to that line.

APPLICATION. *Let* A, *be a point given in the straight line* BC, (*fig.* 20,) *it is required to draw from* A, *a straight line perpendicular to* BC.

Solution. In *AB*, take any point, like *D*, and on the side *AC*, opposite to *AB*, in the line *BC*, make *AE*, equal to *AD*; upon *DE*, construct the equilateral triangle *DEF*; draw *AF*, it will be perpendicular upon *BC*, at the point *A*, as was required.

Proof. In the two triangles *ADF*, and *AEF*, the side *AD*, is equal to *AE*, by construction, as is also the *DF*, to the *EF*; the line *AF*, is common to both triangles, therefore these two triangles have their three sides equal, and the angles opposite to the equal sides equal, therefore the angle *DAF*, is equal to the angle *EAF*; therefore the *AF*, cuts the *BC*, so as to make the angles on the opposite sides of *AF*, equal; therefore they are right angles by the definition, and the *AF*, is perpendicular upon *BC*, as was required.

Scholium I. From this and the preceding Problem, it is evident: that in an equilateral triangle the line from the middle of any one side to the opposite angle is perpendicular to that side; and conversely the perpendicular from any one angular point upon the opposite side will bisect both this angle and the side opposite to it.

Scholium II. From this and Prop. 9, it is evident that the same as stated in the preceding Scholium takes place from the line bisecting the angle included between the equal sides of the isosceles triangle.

§ 12. PROBLEM. From a given point without a given straight line, to draw a perpendicular to that line.

APPLICATION. *Let* A, *be a given point without the straight line* EF, (*fig.* 21,) *it is required to draw from* A, *a line perpendicular to* EF.

Solution. Take any point *B*, on the side *EF*, opposite to that on which *A*, lies; with *AB*, as radius from *A*, as centre, describe a circle, the circumference of which will cut the *EF*, in the points *G*, and *H*, because a part of the circle lies on each side of *EF*; bisect the *GH*, in *I*, and join *AI*, this line will be perpendicular to *EF*, as was required.

Proof. Join *AG*, and *AH;* in the two triangles *AGI*, and *AHG*, there will be the *AG*, equal to *AH*, as radii of the same circle; the *GI*, and *HI*, equal, by bisection of *GH*, and the *AI*, is common to both triangles; they are therefore equal, and the angles opposite to the equal sides are equal; therefore the angles *AIG*, and *AIH*, are equal, being opposite to the equal radii *AG*, and *AH;* they are adjacent angles on the opposite sides of *AI*, therefore they are right angles; and the *AI*, is perpendicular to *EF*, according to definition, as was required.

Corol. The perpendicular is the shortest distance from a given point to a given straight line, because any greater distance produces two intersections with that straight line; and inversely, any line drawn from the same point to any other point of that given line, not lying in the perpendicular, will be greater than this perpendicular.

§ 13. THEOREM. **One straight line meeting another straight line makes with it two angles that are: either both right angles, or together equal to two right angles.**

APPLICATION. *Let* BD, (*fig.* 1,) *be the line which the* CA, *or the* EA, *meets; the angles* CAB, *and* CAD, or EAB, *and* EAD, *are either both right angles, or together equal to two right angles.*

Demonstration. If the two angles are equal, then they are right angles by the definition, (as *CAB*, and *CAD.*) If they are not equal, at the point *A*, draw *AC*, perpendicular to *AB;* the angles *BAC*, and *DAC*, are both equal, and are right angles, the two angles *CAE*, and *EAD*, are together equal to *CAD;* therefore, the three angles *BAC*, *CAE*, and *EAD*, are together equal to the two *BAC*, and *CAD*, these being two right angles, the sum of the three angles is also equal to two right angles; but the angle *BAE* is equal to the two angle *BAC*, and *CAE*, together; therefore, also the two angles *BAE*, and *EAD*, are together equal to two right angles; as was to be demonstrated.

Corol. 1. Any two angles, so situated as in this proposition, are called adjacent angles, and the one forms alternately the supplement of the other to two right angles; therefore, if they are not both right angles, the one must be larger, and the other smaller than the one right angle; that is, the one obtuse, the other acute.

Corol. 2. Any number of lines meeting another line on the same side, and at the same point, always make their sum equal together to two right angles.

§ 14. THEOREM. If from one point in a straight line, two lines are drawn in opposite directions so as to make adjacent angles with that line, which are together, equal to two right angles, these two lines form one continued straight line.

APPLIC. *In figure* 22, *let* A, *be a point in the straight line* AB, *from which two lines are drawn in opposite directions as* AC, *and* AD; *then: if the angles* BAC, *and* BAD, *are together equal to two right angles, the line* CAD, *will be one continued straight line.*

Demonstr. Suppose *AD*, was not in a straight line with *CA*, then another line like *AE*, will be the continuance of the straight line *CA;* therefore, by the last Theorem the two angles *BAC*, and *BAE*, together, will be equal to two right angles; by the supposition also the two angles *CAB*, and *BAD*, are together equal to two right angles; from these two sums take away the common angle *CAB*, there remains the angle *BAE*, equal to *BAD*, the smaller equal to the greater, which is impossible; and as this is the case with any line other than *AD*, this *AD*, is the continuance of the straight line *CA*, as was to be demonstrated.

Corol. Two straight lines that have one part common, are in one continued straight line, because their part common, and that not common, would make angles, with any one line drawn upon their point of meeting, that would be together equal to two right angles.

§ 15. THEOREM. If two straight lines cut one another, the angles which they form, that are opposite to each other at the point of intersection, are equal.

APPLIC. *Let the two straight lines* CB, *and* DE, *cut one another in the point* A, (*fig.* 23,) *then the opposite angles* DAC, *and* BAE, *are equal; and also the angles* DAB, *and* CAE, *are equal to one another.*

Demonstr. The straight line *DE*, makes with the *AB*, at the point *A*, two angles, which are together equal to two right angles; (by Propos. 13,) and the straight line *CB*, makes with the *DA*, at the same point *A*, two angles which are together equal to two right angles; from each of these equal sums take away the common angle *DAB*, the remaining angles *DAC*, and *BAE*, will be equal to one another. The straight line *DE*, being met by the *CA*, at the point *A*, makes the angles *CAD*, and *CAE*, together equal to two right angles; therefore, equal to the sum of the two angles *CAD*, and *DAB*, together; taking from each of these sums the common angle *DAC*, the remaining angles *DAB*, and *CAE*, will be equal; as was to be demonstrated. Two angles situated as are here demonstrated equal, are called *vertical angles.*

Corol. If two or more straight lines cut one another in one point, the sum of all the angles that are made by them around their common point of intersection will always be equal to four right angles.

§ 16. THEOREM. If one of the sides of a triangle is protracted, the exterior angle of this protracted line with the adjacent side, is greater than either one of the interior and opposite angles of the triangle.

APPLIC. *In the triangle* ABC, (*fig.* 24,) *the line* AB, *being protracted, as to* D, *the outward angle* CBD, *will be greater than either of the inward and opposite angles* C, *or* A.

Demonstr. Bisect *CB*, in *E*, draw *AE*, protracted, so as to take in it *EF*=*AE*, join *FB;* the angles *CEA*, and *BEF*, will be equal as vertical angles; they are included by the equal sides just made by the construction; therefore, the two triangles *CAE*, and *BFE*, which they form, are equal, and the angles opposite to the equal sides are equal; that is, *ECA*, and *EBF*, are equal; the outward angle *CBD*, is greater than *CBF*, therefore it is also greater than the angle *C*, of the triangle *ACB*.

Protract also *CB*, to *G*, biseet *AB*, in *H*, draw *CH*, protracted so as to make *HI*=*CH;* then because of the intersecting lines *CI*, and *AB*, the vertical angles *CHA*, and *BHI*, are equal, and they are included by equal sides by the construction; therefore, the triangles *CAH*, and *IBH*, are equal, and the angle *IBH*, is equal to *CAH*, as opposite to the equal sides *CH*, and *HI;* the angle *ABG*, is greater than *ABI;* therefore also, greater than *CAH;* the *ABG*, and *CBD*, are equal, as vertical angles formed by the protraction of the sides of the triangles; therefore also *CBD*, is greater than *CAB*. The outward angle *CBD*, of the triangle *ABC*, is therefore greater than either of the two inward and opposite ones, *A*, or *C;* as was to be demonstrated.

Scholium 1. Any two angles of a triangle are together less than two right angles; because they are less than the sum of the angles made by lines meeting in one point on the same side of one straight line.

Scholium 2. A triangle cannot have more than one angle, a right angle or an obtuse angle, because two such angles together are already equal or greater than two right angles, therefore greater than the sum of the angles formed by the lines meeting another line in one point on one side of the same.

§ 17. THEOREM. To the greater side of a triangle the greater angle is opposite.

APPLIC. *In the triangle* ABC, (*fig.* 25,) *let* AB, *be greater than* AC, *then shall the angle* C, *be greater than the angle* B.

Demonstr. Upon *AB*, the greater line take *AD*, equal to *AC*, the smaller line, join *CD;* the angle *ACD*, will be equal to the angle *ADC*, because *AC=AD;* the angle *ADC*, is outward angle to the triangle *DBC;* therefore, greater than the angle *CBD*, or *CBA*, one of the inward opposite angles of this triangle; therefore, also, *ACD*, and so much more *ACB*, is greater than the angle *B*, of the triangle; but the angle *ACB*, is opposite to the greater side *AB*, and the angle *B*, opposite to the less side *AC;* therefore, to the greater side of a triangle the greater angle is opposite, and to the less the less; as was to be demonstrated.

§ 18. THEOREM. **To the greater angle of a triangle the greater side is opposite.**

APPLIC. *Let the angle at* B, *in the triangle* ABC, (*fig.* 6,) *be greater than the angle* C, *then will the side* AC, *be greater than* AB.

Demonstr. If the side *AC*, is not greater than *AB*, it must be either equal or less than it. Suppose *AC=AB*, then the triangle is isosceles, and the angles *ACB*, and *ABC*, are equal, which is against the supposition that *B*, is greater than *C;* if the side *AC*, is supposed less than *AB*, then by the last proposition the angle *B*, which is opposite to it is also less than the angle *C*, opposite to *AB*, which is also against the supposition; therefore, as the *AC*, can be neither equal nor less than *AB*, it must be greater than it; as was to be demonstrated.

§ 19. THEOREM. **Two sides of a triangle are together greater than the third side.**

APPLIC. *Let* ABC, (*fig.* 26,) *be a triangle, the two sides* AC, *and* CB, *taken together, are greater than the third side* AB.

Demonstr. Protract *AC*, and make *CD=CB*, join *BD;* in the triangle *DBC*, the side *CD*, is equal to *CB*, by construction; therefore, the triangle isosceles, and the angle *CDB*, equal to the angle *CBD;* the angle *ABD*, is greater than *CBD*,

therefore, also, greater than *ADB;* therefore in the triangle *ABD*, the side *AD*, is greater than the side *AB;* but *CD*, being equal to *CB*, the *AD*, is equal to *AC*, and *CB*, taken together; therefore, in the triangle *ABC*, the sum of the two sides *AC*, and *CB*, is greater than the third side *AB;* as was to be demonstrated.

Scholium. The straight line is the shortest line that can be drawn between two given points; for in any other line two straight lines can be drawn from one other point to these points, which will form a triangle, the two sides of which taken together, will always be greater than the third side formed by the straight line between the two given points.

§ 20. PROBLEM. **With three given lines, of which any two taken together are greater than the third, to describe a triangle.**

APPLIC. *Let* H, I, and K, *be three given lines with which the triangle is to be constructed;* (*fig.* 27,) *these lines having the property, that two of them taken together are always greater than the third, as demonstrated in the preceding Theorem to be the case in any three lines forming a triangle.*

Solution. In the indefinite straight line *FG*, lay the three given lines in one continued straight line, as: *DA=H;* the *AB=I;* and the *BC=K;* with *DA*, and from the centre *A*, describe the circle *DEL;* and with the line *BC*, from the centre B, describe the circle *CEM;* the circumferences of these two circles must cut one another, because the sum of their radii is greater than the line *AB*, and the points *D*, and *C*, the other ends of the radii of these circles lie outside of *AB*. From one of the points of intersection *E*, join *EA*, and *EB*, the triangle *EAB*, will be the triangle required, the sides of which will be equal to the three given sides *H*, *I*, and *K*.

Proof. By the construction the three lines *DA*, *AB*, and *BC*, are respectively equal to the three given lines *H*, *I*, and *K*, the line *AE*, is equal to *AD*, as radius of the same circle

DEL; the line *EB*, is equal to the *BC*, as radius of the same circle *CEM;* therefore also the three lines *AE*, *AB*, and *BE*, are respectively equal to the three lines *H*, *I*, and *K;* therefore, the triangle *ABE*, is the triangle required to be constructed.

§ 21. PROBLEM. **At a given point in a given straight line to describe a rectilineal angle equal to a given rectilineal angle.**

APPLIC. *Let* A, *be a point given in the straight line* AI, *at which it is required to describe a rectilineal angle, equal to* CBD, (*figure* 28.)

Solution. In the two sides *BC*, and *BD*, including the given angle, take the two points *E*, and *F*, respectively; join *EF;* from the given point *A*, cut off from *AI*, a line *AG*, equal to *BE*, and with the two lines *BF*, and *EF*, construct upon *AG*, the triangle *AGH*, so that the line *AH*, equal to *BF*, shall end in *A;* draw *AH;* the angle *GAH*, or *IAK*, will be the rectilineal angle laid off from the point *A*, with the line *AI*, equal to the given angle *CBD*.

Proof. The two triangles *FBE*, and *HAG*, are equal by the construction; in the triangle *HAG*, the angle at *A*, is opposite to the side *GH*, which is equal to the line *EF*, in the triangle *FBE*, it is therefore equal to this angle *FBE*, or *DBC*, and laid off from the point *A*, with the line *AI;* as was required to be done.

§ 22. THEOREM. **When two triangles have two angles in the one equal to two angles in the other, each to each, and one of the sides in the one triangle is equal to one of the sides in the other, the triangles are equal in all respects.**

APPLIC. *Let* ABC, *and* DEF, (*figure* 29,) *be two triangles having the angle* A, *equal to the angle* D, *the angle* C, *equal to the angle* F, *and either the side* AC, *equal to* DF; *or the side* AB, *equal to* DE; *the first being between the*

equal angles, the second opposite to one of the equal angles; then shall these triangles be equal in all respects.

Demonstr. In the first case. When the AC, is equal to DF, and the angles A, and C, between which AC, lies, are respectively equal to the angles D, and F, between which DF, lies. Place the triangle ABC, upon the triangle DEF, so that the point A, lie in the point D, the line AC, upon DF; then will the line AB, lie in DE, because the angle A, is equal to the angle D; and the line BC, will lie in FE, because the angle at C, is equal to the angle at F; the point B, must therefore lie in the point E, and the triangle ABC, will be coinciding with the triangle DEF; that is, be equal to it in all respects. For, suppose the point B, did not lie in the point E, then because the angle at D, is equal to the angle at A, the point B, must fall somewhere in the line DE, or its protraction; suppose it fall in H, then joining FH, the angle DFH, will be either smaller or greater than DFE, which is equal to ACB; therefore, DFH, will also be smaller or greater than ACB, which is against the supposition, that it be equal to it. Therefore, when two angles and a side included are equal the triangles are equal in all respects.

In the second case. When the equal sides are opposite to one of the equal angles as AB, opposite to the angle C, equal to DE, opposite to the angle F. Place the triangle ABC, upon the DEF, so that the point A, be in the point D, and the line AB, upon the DE; the point B, will fall in the point E, because AB, is equal to DE; and the point C, will fall in F. For, suppose the point E, did not fall in F, then it must fall somewhere in DF, or its protraction, because the angle A, is equal to the angle D, suppose it lie in I; join IE, then the angle DIE, is greater than DFE, as outward angle of the triangle FIE; but the angle C, is equal to DFE, therefore DIE, is also greater than C, which is against the supposition.

If, in either case, the point is supposed to fall in the protraction of the side of the triangle, the demonstration will be exactly the same, except that the angles which become greater

or smaller, would then become smaller or greater than the angle to which they should be equal.

Corol. Two angles and one side, any how situated to one another, determine a triangle.

§ 23. THEOREM. **If two straight lines are cut by a third straight line, so that this line makes with them: alternate inward angles that are equal; or the inner and outer angle, with the two straight lines, on the same side of the cutting line, equal, respectively; or the two inward angles on the same side of the cutting line, together equal to two right angles these two straight lines are parallel.**

APPLIC. *Let the two straight lines* AB, *and* CD, (*fig.* 30,) *be cut by the third straight line* EF; *so that either* AGH=GHD, *or* EGB=GHD; *or that* BGH, *and* GHD, *together, shall be equal to two right angles; then the lines* AB, *and* CD, *are parallel; that is, protracted indefinitely, they will never meet* (*definition* 25.)

Demonstr. 1. When the alternate inward angles *AGH*, and *GHD*, are equal, then the protractions of the two straight lines *AB*, and *CD*, will never meet.

Antithesis. Suppose that the straight lines *AB*, and *CD*, protracted, do meet, as in the point *I;* then the figure *GHI*, will be a triangle, (defin. 19,) and the *GA*, is the protraction of the side *DI*, of this triangle; therefore, the angle *AGH*, is an outward angle of this triangle, and greater than *GHD*, one of the inward and opposite angles; but by the supposition, the angle *AGH=GHD;* this is, therefore, contradictory, and the *AB*, and *CD*, do not meet in their protraction; that is, they are parallel, according to the definition.

2. When the inward angle *GHD*, of the cutting line *EF*, with the *CD*, is equal to the outward angle *EGB*, of the line *AB*, with the cutting line *EF*, on the same side of it: then the lines *AB*, and *CD*, are parallel; that is, their protractions never meet.

Anti-thesis. Suppose as before, that the straight lines *AB*, and *CD*, do meet in *I*, their protraction, then, *GHI*, is a triangle, and the angle *EGB*, is an outward angle to this triangle; therefore, greater than the inward and opposite one *GHI*; but by the supposition, these two angles are equal, which is contradictory; therefore, the protractions of *AB*, and *CD*, do not meet; that is, they are parallel.

3. When the two inward angles *BGH*, and *DHG*, on the same side of *EF*, are together equal to two right angles, the *AB*, and *CD*, are parallel.

Anti-thesis. Suppose again that the straight lines *AB*, and *CD*, do meet in *I*, their protraction; then *GHI*, is again a triangle, and (by Prop. 16, Schol. 1,) the two angles *IGH*, and *IHG*, are together less than two right angles; but by the supposition these two angles are equal to two right angles, which is again contradictory; therefore, the protracted straight lines *AB*, and *CD*, do not meet; they are, therefore, parallel, according to the definition.

What has been proved here for the point *I*, is equally applicable to any point on either side of the cutting line *EF*; and what has been said of the angles quoted, applies equally to the angles equally situated on the opposite sides of either of the lines; therefore it is general; and when two straight lines are cut by a third, so as to have the alternate inner angle, or the inner and outer angle on the same side of the cutting line, equal, or the two inner angles on the same side of the cutting line equal to two right angles, these lines are parallel to one another, or they never meet.

Corol. When the parallel lines are cut perpendicularly by the third line, the part of the cutting line between the parallels is said to be the distance between these parallels.

§ 24. THEOREM. **When a straight line cuts two parallel straight lines, it makes the alternate inward angles with these parallels equal; the inner and outer angle on the same side of the cutting line equal; and**

the two inner angles on the same side, together, equal to two right angles.

APPLIC. *Let* AB, *and* CD, (*fig.* 30,) *be two parallel straight lines, which are cut by the third straight line* EF, *then will be: the angle* AGH=GHD; *the angle* BGH=GHC, *as alternate angles; the angle* EGB=GHD; *the* EGA=GHC; *the* CHF=AGH, *and the* BGH=DHF, *as outer and inner angles on the same sides of the cutting line; and the two angles* BGH, *and* DHG, *will be together, equal to two right angles, as will also be the two angles* AGH, *and* GHC.

Anti-thesis 1. Suppose the angle *AGH*, be not equal to *GHD*, but greater than it, the angles *AGH*, and *BGH*, are together, equal to two right angles; therefore, the angles *BGH*, and *GHD*, are together, less than two right angles; the sides *GB*, and *HD*, protracted, will therefore form a triangle, and will cut each other, suppose in *I*, but they are supposed to be parallel: that is, never to cut in their protraction; therefore, the angle *AGH*, is not greater than *GHD;* it is also not less, because it would then make with the angle *GHC*, angles which together, would be less than two right angles; and the lines *AB*, and *CD*, would meet on that side of the cutting line *EF;* as it can be neither less nor greater, it must be equal; and the same will be the case with the other alternate inner angles; as was to be demonstrated.

2. Suppose the angle *EGB*, not equal to *GHD*, but greater than it, then again *EGB*, and *BGH*, make together two right angles, therefore *BGH*, and *GHD*, are together *less* than two right angles, and the lines *GB*, and *HD*, or their protraction, form two sides of a triangle; that is, they cut one another, and the lines *AB*, and *CD*, of which they are parts, are not parallel, which is against the supposition. If the angle *EGB*, was smaller than *GHD*, the same reasoning would apply to the other side of the cutting line *EF;* and what applies to these two inner and outer angles, applies to all the others.

3. Suppose the angles *BGH*, and *GHD*, together, not equal to two right angles, but less than it, then the two lines *GB*,

and *HD*, or their protraction, would be the sides of a triangle, of which they are the angles, or, the lines would cut each other, which is against the supposition; and if they were together greater than two right angles, the same thing would take place on the other side of the straight line *GH*.

Therefore, when two straight lines are parallel and are cut by a third straight line they make: the alternate inward angles equal; the outer and inner angles on the same side of the cutting line, in the same direction, equal; and the two inner angles on the same side of the cutting line together equal to two right angles.

Corol. 1. Through one point outside of a straight line only one line can be drawn parallel to that line, because any other would form other angles than proved to result from parallel lines.

Corol. 2. When one of two parallel straight lines is cut by a third straight line, the other will also be cut by the same, or the protractions; because they must cut one another under the same angle as the first parallel.

§ 25. PROBLEM. Through a given point outside of a given straight line to draw a line parallel to the same.

APPLIC. *Let* A, *be a point outside of the straight line* BC, (*fig.* 31,) *it is required to draw through* A, *a straight line parallel to* CB.

Solution. Through the point *A*, draw any line as *AD*, cutting the straight line *BC*, in the point *D*; at the point *A*, make with the line *AD*, an angle *EAD*, equal to *ADC*, on the side of *AD*, opposite to *ADC*; the line *EAF*, will be drawn through the point *A*, parallel to *BC*.

Proof. The angle *EAD*, being equal to *ADC*, makes the alternate inner angles between the *EF*, and *BC*, with the cutting line *AD*, equal; therefore, these lines parallel to one another, according to the preceding proposition.

§ 26. THEOREM. If two straight lines are each parallel to a third straight line, they are parallel to one another.

APPLIC. *If the two straight lines* AB, *and* CD, *are each of them parallel to* EF, (*fig.* 32,) *they are also parallel one to the other.*

Demonstr. Draw the straight line *GI*, cutting the three straight lines *AB*, *CD*, and *EF*. The two straight lines *AB*, and *EF*, being parallel, the angles *BGH*, and *GHE*, are equal, as inward and opposite angles; the *CD*, and *EF*, being also parallel, the angle *GHE*, is equal to the angle *HIC*, as inner and outer angle on the same side of the cutting line *GI;* therefore, the two angles *BGI*, and *GIC*, are equal to one another, and they form alternate angles of *AB*, and *CD*, with the cutting line *GI;* therefore, the lines *AB*, and *CD*, are parallel; as was to be demonstrated.

§ 27. THEOREM. **If one side of a triangle be protracted, the outward angle formed by the protraction, and the adjacent side of the triangle is equal to the sum of the two inward and opposite angles of the triangle; and all the three angles of a triangle: are together equal to two right angles.**

APPLIC. *The side* AB, *of the triangle* ABC, (*fig.* 33,) *being protracted towards* D, *the outer angle* CBD, *will be equal to the two inward and opposite angles* A, *and* C, *taken together; and the three inward angles* A, B, *and* C, *of the triangle will be, together, equal to two right angles.*

Demonstr. Through the angular point *B*, of the triangle draw the *BE*, parallel to *AC;* the *ABD*, cutting the two parallel straight lines *AC*, and *BE*, makes the angle *EBD*, equal to the angle *CAB*, of the triangle, as inner and outer angle; the line *CB*, cutting the parallel straight lines *BE*, and *AC*, makes the angle *CBE*, equal to *ACB*, as alternate inner angles; therefore, the two angles *CBE*, and *EBD*, are together, equal to the two angles *A*, and *C*, of the triangle; the outward angle *CBD*, is equal to the two angles *CBE*, and *EBD*, together; therefore, equal to the two inward and opposite angles *A*, and *C*, of the triangle; that is, the outward angle of the protracted side of the triangle with the adjacent one,

is equal to the sum of the two inward and opposite angles. If to the two angles *A*, and *C*, of the triangle, the third angle *B*, be added, the sum will be equal to all the angles of the triangle; and likewise, if to the sum of the two angles *CBE*, and *EBD*, the same angle *B*, of the triangle be added, these sums will be equal, and also equal to all the angles formed by the lines meeting the *AD*, in the point *B*, on one side of the line, which have been proved to be together, equal to two right angles; therefore, also, the sum of the three angles of a triangle is equal to two right angles; as was to be demonstrated.

Corol. When two angles of a triangle are given, the third angle is the difference between their sum and two right angles, and alternately the sum of any two angles of a triangle is equal to the difference between the third angle and two right angles.

§ 28. THEOREM. In any rectilineal figure the sum of all the inner angles together is equal to twice as many right angles as the figure has sides, wanting four right angles.

APPLIC. *In the rectilineal figure* ABDEFG, (*fig.* 34,) *the sum of all the inner angles* A, B, D, E, F, *and* G, *is equal to twice as many right angles as the figure has sides, wanting four right angles.*

Demonstr. From any point *C*, within the figure draw straight lines to all the angular points, there will be as many triangles formed as there are sides in the rectilineal figure. Each of these triangles will have the sum of its angles equal to two right angles; therefore, they will all together have twice as many right angles as the figure has sides; the angles of these triangles, which are about the point *C*, are together equal to four right angles; taking these away from the sum of all the angles of the triangles, there remain all the angles of the poligon, their sum will therefore be equal to twice as many right angles as the figure has sides wanting these four right angles; as was to be demonstrated.

Corol. 1. When all the sides or all the angles of such a poligon are equal, it is called a regular poligon, such is the equilateral triangle, the square, and so on for the greater numbers.

CHAPTER III.

Of the equality of Triangles with quadrilateral and other rectilineal Figures.

§ 29. THEOREM. **When two parallel straight lines cut two other parallel straight lines, the opposite sides and angles of the resulting quadrilateral figure are equal; a diagonal drawn between two opposite angles, divides it into two equal triangles; and all the inner angles of the quadrilateral figure are together, equal to four right angles.**

APPLIC. *Let* (*fig.* 35,) *the two parallel lines* AB, *and* CD, *cut the two parallel lines* EF, *and* GH; *the quadrilateral figure* IKML, *will have the angle at* L, *equal to the angle at* K; *the angle* I, *equal to the angle* M; *the side* IL=KM, *and* IK=LM; *the diagonal* LK, *will cut it into the two equal triangles* LIK, *and* LMK; *and all the four angles* I, K, L, *and* M, *together, are equal to four right angles. The quadrilateral figure thus resulting is called a* Parallelogram.

Demonstr. The parallel straight lines *AB*, and *CD*, being cut by *EF*, make the inward and opposite angles *AIL*, and *ILM*, equal; the parallel straight lines *EF*, and *GH*, being cut by *AB*, the angles *AIL*, and *IKM*, are equal as outer and inner angle on the same side of *AB*; therefore, the angles *ILM*, and *IKM*, are each equal to *AIL*; therefore, they are equal to one another. In the same manner is proved, that *LIK*, is equal to *LMK*; therefore, the opposite angles in the quadrilateral figure *IKML*, are equal.

The diagonal *LK*, cutting the two parallel straight lines, *EF*, and *GH*, makes the angles *ILK*, and *LKM*, equal; the diagonal being common to the two triangles *ILK*, and *KLM*, these two triangles have two angles in the one equal to two

angles in the other; for, the angles *I*, and *M*, have been proved equal, and the side *LK*, opposite to these angles, is the same in both triangles; these two triangles are, therefore, equal in all respects; the sides *IL*, and *KM*, are equal, being opposite to equal angles; and also, *IK*, and *LM*, are equal, as opposite to equal angles; these lines are also opposite sides in the parallelogram *IKML;* therefore, the opposite sides of the parallelogram are equal. The sum of all the angles of the two triangles *IKL*, and *LKM*, together, are equal to four right angles; therefore, also, all the angles of the parallelogram are together, equal to four right angles; as was to be demonstrated.

Corol. 1. The lines which join equal and parallel lines, are equal; or parallels between parallels are equal.

Corol. 2. When two opposite angles of a Parallelogram are together equal to two right angles, each of them is one right angle.

Corol. 3. When one of the angles of a parallelogram is a right angle, all the others are right angles, and the figure is called a *rectangular parallelogram.*

Corol. 4. When one angle of the parallelogram is a right angle, and the two sides adjacent to it are equal, all the angles are right angles, and all the sides are equal; because all the opposite sides and angles are equal. This figure is called a *Square.*

Corol. 5. The diagonal of the square divides it into two isosceles right angled triangles.

§ 30. THEOREM. **When the opposite sides and angles of a quadrilateral figure are equal, these opposite sides are parallel, and the figure is a Parallelogram.**

APPLIC. *In (figure* 35,) *the opposite sides* IL, *and* KM, *and the* IK, *and* LM, *in the quadrilateral figure* IKLM, *being equal; and the opposite angles* IKM, *and* ILM, *also equal; then the line* IL, *is parallel to* KM, *and the* IK, *parallel to* LM; *the whole figure* IKLM, *is therefore a Parallelogram.*

Demonstr. Draw the diagonal *LK;* in the two triangles *IKL*, and *LKM*, the two sides *IL*, and *IK*, are equal to the two sides *KM*, and *LM*, each to each, and the diagonal *LK*,

is common to both triangles *IKL*, and *MLK;* therefore, these two triangles are equal, and the angles opposite to equal sides are equal; that is, the angle *ILK*, is equal to *LKM;* therefore, the straight lines *IL*, and *KM*, which this diagonal cuts, are parallel. In the same manner it is proved, that *IK*, and *LM*, are parallel; therefore, the opposite sides of the figure *IKML*, are parallel, and the figure is a Parallelogram, having all the properties assigned to it in the preceding proposition, with their consequences; as was to be demonstrated.

Corol. If two sides adjacent to a given angle of a parallelogram are given, the parallelogram is given.

§ 31. THEOREM. **Parallelograms which have two opposite sides in the one equal to two opposite sides in the other, and these lines in the same parallel lines, are equal to one another.**

APPLIC. *Let* ABCD, *and* EFGH, (*figure* 36,) *be two parallelograms which have the sides* AB, *and* CD, *and* EF, *and* HG, *all equal, and the lines* AB, and EF, *in the same straight line* AF, *parallel to* CG, *in which the* CD, *and* HG, *lie; these two parallelograms are equal.*

Demonstr. Join *CE*, and *DF;* the sides *AB*, and *EF*, of the two parallelograms *AD*, and *EG*, being equal by the supposition, if to each of them the line *BE*, be added, the straight lines *AE*, and *BF*, are equal; the line *AC=BD*; the angle *CAE*, is equal to the angle *DBF*, because the two parallels *AC*, and *BD*, are cut by the straight line *AF;* therefore, the two triangles *ACE*, and *BDF*, are equal, and their third sides *CE*, and *DF*, are equal; therefore, the figure *CEFD*, is a parallelogram. From the whole figure *AFDC*, take away the triangle *BFD*, there remains the parallelogram *ABCD;* and from the same whole figure take away the triangle *AEC*, there remains the parallelogram *ECDF;* the two triangles being equal the two remaining parallelograms are equal also. The side *HG*, in the parallelogram *EFGH*, is equal to the side *CD*, in the parallelogram *ABCD*, by supposition; to each of

these the line DH, being added, the sums CH, and DG, are equal; the FG, is equal to EH, and $CE=DF$, as just proved; the two triangles CEH, and FDG, are therefore equal. From the whole figure $ECGF$, take away once the triangle ECH, and the other time the triangle DFG, there remains in the first case the parallelogram $EFGH$, and in the second the parallelogram $EFDC$, that must be equal to one another, as equals subtracted from equals are equal. The two parallelograms AD, and EG, are therefore each equal to the parallelogram $EFDC$; therefore, they are equal to one other, as was to be demonstrated.

Scholium 1. If lines be drawn perpendicular between two parallels, at the two extremities of the sides of any parallelogram, the rectangular parallelogram resulting, will be equal to that parallelogram; and to any parallelogram upon an equal base, and between the same parallels, under any angle whatsoever, as the above Demonstration applies to all equally.

Scholium 2. The perpendicular distance between two parallel sides of a parallelogram is called its altitude; thence, the usual expression is: parallelograms upon *equal sides* or bases, and of the same *altitude*, are equal.

Scholium 3. Every parallelogram being divisible by the diagonal into two equal triangles, also triangles of *equal bases* and *altitude*, are equal.

Scholium 4. For the same reason also the triangle is half the parallelogram, of equal base and altitude.

§ 32. PROBLEM. To describe a parallelogram equal to a given triangle, and which shall have one of its angles equal to a given rectilineal triangle.

APPLIC. *Let* BCD, (*figure 31,*) *be the given triangle;* A, *the given rectilineal angle: it is required to describe a parallelogram, as* EBFG, *equal to the given triangle, and having the angle at* B, *equal to the angle* A.

Solution. Bisect the line BC, in E; at the point B, make the angle CBG, equal to the given angle A; through E, draw EF, parallel to BG; and through the angular point D, of the

given triangle, draw the *GDF*, parallel to *BC*, so as to cut the two parallel lines *BG*, and *EF*, in *G*, and *F*; the figure *BEFG*, will be the parallelogram required, equal to the triangle *BCD*, and having the angle *B*, equal to the given angle *A*, as required.

Proof. Join *DE*; the two triangles *BDE*, and *DEC*, are equal, being upon equal bases, and between the same parallels; the triangle *BED*, stands with the parallelogram *BEFG*, upon the same base *BE*, and between the same parallels *BE*, and *GF*; it is therefore half that [illegible]; the two triangles *BDE*, and *EDC*, together, are also double the one of them; therefore, equal to the parallelogram *BEFG*; therefore, this parallelogram is constructed as was required, equal to the given triangle and the angle at *B*, equal to the given angle *A*.

Scholium. As the angle given in this proposition may be a right angle, we can make a rectangular parallelogram equal to a given triangle; and this rectangle will be equal to one having half the base of the triangle, and the same altitude.

§ 33. THEOREM. **When through one point in the diagonal of a parallelogram, straight lines are drawn parallel to the different sides of the parallelogram, those parallelograms which they form are equal, through which the diagonal does not pass.**

APPLIC. *If in the parallelogram* ABCD, (*fig.* 38,) *through the point* EI, *in the diagonal* AC, *the* FG, *and* KI, *are drawn parallel to* AD, *and* AB; *the parallelograms* FBIE, *and* KEGD, *through which the diagonal* AC, *does not pass, are equal to one another.*

Demonstr. The two triangles *ABC*, and *ADC*, are equal, each of them being half the parallelogram *ABCD*; the lines *KI*, and *FG*, divide this parallelogram into four parallelograms, having one angle common with the larger triangle; the part *EC*, of the diagonal *AC*, cuts the parallelogram *EGCI*, into two equal triangles *EGC*, and *EIC*; the part *AE*, of the same diagonal divides also the parallelogram *AFEK*, into two

equal triangles *AFE*, and *AKE*, therefore the sum of the two triangles *EIC*, and *AFE*, is equal to the sum of the two triangles *EGC*, and *AKE;* taking the first sum from the triangle *ABC*, and the second sum from the triangle *ADC*, there remains the parallelogram *EFBI*, equal to the parallelogram *EKDG;* these parallelograms are those through which the diagonal does not pass; therefore the parallelograms that lie at the side of the diagonals, or through which the diagonal of a larger parallelogram does not pass, are equal to one another.

§ 84. PROBLEM. To describe a parallelogram equal to a given triangle, that shall have one of its sides equal to a given straight line, and one of its angles equal to a given rectilineal angle.

APPLIC. *Let* CDE, *be the given triangle;* A, *the given straight line, and* B, *the given angle; it is required to describe a parallelogram equal to the triangle* CDE, *having one side equal to* A, *and one angle equal to the angle* B, (*fig.* 39.)

Solution. Describe the parallelogram *IHPK*, equal to the given triangle *CDE*, and having one of its angles equal to the given angle *B;* (as shewn in § 32.) Protract the *KH*, and make *HG*, equal to the straight line *A*; through *G*, draw *FGN*, parallel to *IH;* protract *PI*, until it intersects the *FGN*, in the point *F*, which will form the parallelogram *FGHI;* draw the diagonal *FH*, of this parallelogram, and protract it; protract also the *PK*, until it intersects the protracted diagonal in the point *L;* through *L*, draw a line parallel to *KG*, and *FP*, until it intersects the *FN*, in *N;* protract also the *IH*, until it intersects the *LN*, in *M;* the parallelogram *GHMN*, will be the parallelogram required, equal to the triangle *CDE*, having the angle *GHM*, equal to the angle *B*, and the side *HG*, equal to the given line *A*.

Proof. The parallelogram *PIHK*, is equal to the triangle *CDE*, by construction, having the angle *IHK*, equal to the given angle *B*, the quadrilateral figure *FPLN*, is a parallelogram, because the *PL*, as protraction of *PK*, is paral-

lel to *FN*, and *IH;* the diagonal *FL*, divides it into two equal triangles; in like manner *HGNM*, is a parallelogram, and is equal to the parallelogram *IHKP*, because these two parallelograms lie about the diagonal of the larger parallelogram *FPLN;* the angle at *H*, is equal to the angle at the same point in the parallelogram *IK*, as a vertical angle, and the side *HG*, has been made equal to the given line *A;* therefore, the parallelogram *HN*, and the triangle *CDE*, are both equal to the parallelogram *IK;* therefore, the parallelogram *HN*, is equal to the triangle *CDE*, and described upon the given side *A*, with one angle equal to the given angle *B;* as was required to be done.

Scholium 1. By this Problem, we can, therefore, make a rectangle that shall be equal to a given triangle, and having one side equal to a given straight line.

Scholium 2. As any rectilineal figure can be decomposed into triangles, and by this Problem a rectangle can be described equal to a triangle, with one of its sides being given: we can also make a rectangle, with a given side, equal to any rectilineal figure, by the continued addition, under the same constant side, of the rectangles equal to each of the triangles into which the rectilineal figure is decomposed.

§ 35. PROBLEM. To describe a square, the sides of which are equal to a given straight line.

APPLIC. *It is required to describe a square, the sides of which shall be equal to the straight line* E, (*fig.* 40.)

Solution. Describe a right angle *FAG*, upon the two lines *AG*, and *AF*, take the lines *AD*, and *AB*, equal to *E;* through *D*, and *B*, draw the lines *DC*, and *BC*, parallel to *AF*, and *AG;* they will intersect one another in *C*, and the quadrilateral figure *ABCD*, will be the square required, the sides of which are equal to the straight line *A*.

Proof. The sides *AD*, and *AB*, are each equal to the straight line *E*, by construction, the angle at *A*, a right angle; the *DC*, and *BC*, being drawn parallel to *AB*, and *AD*, are also equal to these lines, as opposite sides of the paralello-

gram; the angle *A*, being a right angle, all the other angles are right angles; and the sides *AB*, and *AD*, being equal to the given straight line *E;* all the sides of the quadrilateral are equal, and the figure is a square described with the straight line *E;* (definition 24,) as was required to be done.

§ 36. THEOREM. **In a right angled triangle, the square described upon the side opposite to the right angle, is equal to the sum of the squares described upon the other two sides.**

APPLIC. *Let* ABC, (*fig.* 41,) *be a triangle, right angled in* A, *the square described upon* BC, *the line opposite to the angle* A, *shall be equal to the sum of the squares described upon* AB, *and* AC, *the two other sides.*

Demonstr. Describe upon the three sides respectively the squares *BH*, *BE*, and *CF;* then the line *EAC*, will be one continued straight line parallel to *DB*, because it makes with the *AB*, at the point *A*, two angles together, equal to two right angles; and for the same reason, *BAF*, is one continued straight line, and parallel to *CG;* the angles at *A*, being in each case formed by the right angle of the triangle, and the one of the right angles of the respective square. Through the point *A*, draw *AK*, parallel to *BI*, and *CH;* join *CD*, and *AI*, and also *BG*, and *AH*. The triangle *BDC*, and the square *BDEA*, have the side *BD*, common, and are between the same parallels *BD*, and *EC;* the triangle *BDC*, is, therefore, half the square *BDEA;* the triangle *BAI*, and the parallelogram *BIKL*, have the same base *BI*, and are between the same parallels *AK* and *BI;* the triangle *ABI*, is, therefore, half the parallelogram *BK*. In the two triangles *CBD*, and *ABI*, the angle *DBC*, is equal to the angle *ABI*, being each of them equal to the sum of one right angle of one of the squares, and the angle *ABC*, of the triangle; the sides *BD*, and *BC*, which include the angle *DBC*, are equal respectively, to the sides *AB*, and *BI*, including the equal angle *ABI;* in the two triangles *DBC*, and *ABI*, we have, there-

fore, two sides and the included angle in the one, equal to two sides and the included angle in the other; therefore, these two triangles are equal to one another; therefore, also, their doubles are equal; that is, the square *ABDE*, is equal to the parallelogram *BLKI*. Exactly in the same manner it is proved: that the triangle *BCG*, is equal to the triangle *ACH*, and the square *ACGF*, equal to the parallelogram *CHKL;* the two parallelograms *BK*, and *KC*, are, therefore, together, equal to the two squares *EB*, and *CF*, taken together; but the two parallelograms *BK*, and *KC*, together, form the square *BH*, described upon the side of the right angled triangle; that is, opposite to the right angle; therefore, the sum of the two squares *BE*, and *CF*, described upon the two other sides *AB*, and *AC*, including the right angle, are, together, equal to the square upon *BC*, opposite to the right angle; as was to be demonstrated.

Scholium 1. By the preceding will be found the side of a square, which shall be equal to the sum of two squares, the sides of which are given: by a right angle being formed, upon the two sides, of which the two sides of the given squares will be laid off; the line joining the extremities of these lines, will be equal to the side of the square, that will be equal to the sum of the two other squares.

Scholium 2. The side of the square which shall be equal to the sum of any number of squares, may therefore also be found by continuing the preceding operation, between the line obtained at first, and the other lines successively, until they have all been employed in the operation.

Scholium 3. The side of the square equal to the difference of two squares, will therefore also be obtained by taking on the one side, including the right angle, a line equal to the side of the smaller square, and with the side of the larger square, describing a circle, the intersection of which with the other line about the right angle, will cut off a line equal to the side of the square, that will be equal to the difference of the two given squares.

§ 87. THEOREM. When the sum of the squares of the two sides of a triangle, is equal to the square of the

third side, the angle of the triangle contained between the two smaller sides, is a right angle.

APPLIC. *Let* ABC, (*fig.* 42,) *be a triangle, in which the square upon* AB, *is equal to the sum of the squares upon* AC, *and* BC, *the angle at* C, *will be a right angle.*

Demonstr. Upon *CB*, and at the point *C*, make *BCD*, equal to a right angle, and *CD*, equal to *CA*, join *DB;* the triangle *BCD*, being right angled at *C*, the sum of the squares upon *CB*, and *CD*, is equal to the square upon *DB;* the *CD*, being equal to *CA*, the sum of the squares upon *CB*, and *CD*, is equal to the sum of the squares upon *CB*, and *CA;* but the sum of the squares upon *CB*, and *CD*, is equal to the square upon *BD*, (by § 36,) and the sum of the squares upon *CB*, and *CA*, is equal to the square upon *AB*, by the supposition; therefore, the square upon *DB*, is equal to the square upon *AB;* therefore, the *DB*, is equal to *AB*, and the two triangles *ACB*, and *DCB*, have their three sides equal each to each, and the angles *ACB*, and *DCB*, are equal, being opposite to the equal side *AB*, and *DB;* the angle *DCB*, is a right angle by construction; therefore, also, *ACB*, is a right angle; as was to be demonstrated.

Scholium. A right angle may be constructed according to this proposition, by forming a triangle such: that the sum of the squares of two of its sides be equal to the square of the third side. Thus will be for instance a triangle, the sides of which would be 3, 4, and 5, for $9+16=25$, gives to the square of these numbers the property required; and so would their equimultiples do the same.

CHAPTER IV.

Of the Circle, and the straight lines referring to the Circle.

§ 38. THEOREM. A straight line cannot cut the circumference of a circle in more than two points.

APPLIC. *Let the circle* ABDE, (*fig.* 43,) *the centre of which is* F, *be cut by the straight line* AC, *the circumference of the circle shall be cut by it only in the two points* A, *and* B.

Demonstr. If possible, suppose the straight line *AB*, cut the circumference of the circle also in another point, as *D*; join *BD*, then *ABD*, is a straight line; from the centre of the circle *F*, draw *FG*, perpendicular upon *AB*, and *FH*, perpendicular upon *DB*; join also *FB*, which meeting the supposed straight line *ABD*, in the point *B*, must make with it angles on one side that are together equal to two right angles, that is the angles *ABF*, and *FBD*, are together equal to two right angles. In the quadrilateral figure *FGBH*, the four angles together are equal to four right angles; the angles *FGB*, and *FHB*, are right angles by construction, therefore together equal to two right angles; and the angles *GFH*, and *ABD*, together, must be equal to two right angles; therefore, the *GFH*, must be less than two right angles, but the *GBH*, which is the sum of *GBF*, and *FBH*, is equal to two right angles, by the supposition; therefore *GBH*, would be at the same time equal and less than two right angles, which is impossible. Therefore, the straight line *AB*, does not cut the circumference of the circle *ABDE*, in the point *D*, or any other point except *A*, and *B*; as was to be demonstrated.

Coral. The two straight lines joining three points in the circumference of a circle make together an angle less than two right angles.

§ 39. PROBLEM. To find the centre of a circle.

APPLIC. *Let* ABDI, (*fig.* 44,) *be the circumference of a circle, it is required to find the centre of it.*

Solution. Draw any line *AB*, cutting the circumference of the circle in *A*, and *B*; bisect *AB*, in *E*, and from *E*, draw the *EG*, perpendicular to *AB*; from *B*, draw any other line in the circle as *BD*, bisect it in *F*, and from it draw *FH*, perpendicular to *BD*; these two perpendiculars will intersect one another, because the angle *ABD*, is less than two right angles; their point of intersection *C*, is the centre of the circle, which is required to be found.

Proof. Join *CA*, *CB*, and *CD*; in the two triangles *AEC*, and *BEC*, the sides *AE*, and *EB*, are equal by the bisection, the *CE*, is common to these two triangles, and the angles which these equal sides include at *E*, are right angles, therefore equal; therefore, the two triangles are equal, and the sides *AC*, and *BC*, opposite to the right angles at *E*, are equal. Exactly in the same manner is proved, that in the two triangles *CBF*, and *CDF*, the sides *CB*, and *CD*, are equal, therefore the three sides *CA*, *CB*, and *CD*, are all equal to one another; and any number of points may be taken in the circumference of the circle, between which straight lines being drawn and bisected, the perpendiculars drawn at their point of bisection, will meet in the same point *C*, as they would always form equal triangles with the two parts of the bisected line and the perpendicular, and all their sides opposite to the right angles become equal. This point *C*, of their common intersection will therefore be equidistant from any point of the circumference of the circle; that is, it will be the centre of it, and the *CA*, *CB*, *CD*, and so forth, will be the equal radii of the circle.

Corol. 1. When more than two equal straight lines can be drawn from a point within the circle to the circumference, that point is the centre of the circle.

Corol. 2. Three points determine the circumference of a circle, for the continuance of the same circle has all the radii equal.

Corol. 3. The line drawn perpendicular to the middle of a chord, passes through the centre of the circle.

Corol. 4. The line from the centre of the circle perpendicular to the chord bisects that chord, and the angle at the centre; for the two triangles upon the half chords being equal in all respects, have the angles at the centre equal.

§ 40. PROBLEM. **Through three given points, that are not in one straight line to draw the circumference of a circle.**

APPLIC. *The three points* A, B, *and* D, (*fig.* 44,) *which do not lie in one straight line, being given, to draw the circumference of a circle through them.*

Solution. Join *AB*, and *BD*, and bisect them in *E*, and *F*; draw the perpendiculars *EG*, and *FH*; these will cut one another, because the lines *AB*, and *BD*, make together an angle less than two right angles; the point *C*, where these lines intersect one another, will be the centre of a circle, thus: that taking *CB*, as radius, and describing a circle, this will pass through the three given points, *A*, *B*, and *D*.

Proof. Join *CA*, *CB*, and *CD*; in the two triangles *AEC*, and *BEC*, the angles *CEA*, and *CEB*, are right angles, the side *EA*, is equal to *BE*, and the *CE*, is common to both triangles; therefore, the third sides *AC*, and *BC*, are also equal; by the same reasoning upon the triangles *BFC*, and *DFC*, the *CD*, is proved equal to *CB*; therefore, the three lines *CA*, *CB*, and *CD*, are all equal, and the circumference of a circle drawn with either one of them, will pass through the two other points; as was required.

Corol. 1. If only part of the circumference of a circle is given, the circle can be completed by taking three points in the part given, and determining the centre by the above Problem.

Corol. 2. To any triangle a circle can be circumscribed by this Problem, as every two sides of the triangle constitute the junction between points in the circumference of a circle, upon which the Problem applies. Thence three points determine also a circle as well as a triangle, provided they be not in the same straight line.

(*Remark.*) By § 38, it has been proved, that three points cannot lie in one straight line, and at the same time in the circumference of one circle.

§ 41. THEOREM. If two circles cut one another, or touch one another internally, they have not the same centre.

APPLIC. *Let* ABF, *and* ADE, (*fig.* 45,) *be two circles, cutting one another, or touching one another internally in the point* A, *they shall not have the same centre.*

Demonstr. If possible, suppose C, to be a common centre to the two circles A, their point of intersection, or contact; join CA, and draw from C, any other line cutting the two circles in the points B, and D; then CB, is equal to CA, as radius of the same circle ABF; and also CD, is equal CA, as radius of the same circle ADE; therefore, the two radii CB, and CD, being both equal to CA, they are equal to one another, that is, the part is equal to the whole, which is impossible; therefore, the point C, is not a common centre to the two circles, and no other point can be a common centre, because the same reasoning applies to any point whatsoever; therefore, two circles that cut one another, or touch one another internally, have not the same centre.

§ 42. THEOREM. The circumference of one circle cuts the circumference of another circle in no more than two points.

APPLIC. *Let the circumferences of the two circles* ADF, *and* ADE, (*fig.* 46,) *cut one another, they will cut one another only in two points, as* A, *and* D, *and in no other.*

Demonstr. Draw the straight line BC, joining the centres of the two circles; draw also AB, and AC, from one of the points of intersection to the two centres. If they were to cut one another in any other point, as E, then because CE, would be equal to CA, and at the same time BA, equal to BE, as radii of their corresponding circles $AEGD$, and ADF, there would be on the same side of BC, two triangles having their sides ending in the same points of the BC, equal to one another, which is impossible. The same applies on the other side of

BC, in relation to any point other than *D*, the other point of intersection of the two circles; therefore, the circumferences of the two circles cannot cut one another in more than two points.

§ 43. THEOREM. When the circumferences of two circles touch one another, internally or externally, the two centres and the point of contact are in the same straight line.

APPLIC. *The two circles* AEG, *and* ADF, (*figure* 47,) *touch one another internally or externally in* A, *the line joining their centres shall pass through the point of contact.*

Demonstr. If the line joining the centres does not pass through the point of contact *A*, let the line *BC*, be drawn between the centres, which does not pass through the point *A*; *C*, being the centre of the circle *ADF*, and *B*, the centre of the circle *AEG*; join *CA*, and *AB*, then:

I. When the two circumferences touch one another internally, the *CA*, and *CB*, will be together, greater than *BA*, as two sides of the triangle *ABC*; the *CD*, is equal to *CA*, as radius of the same circle; therefore, also, *BC*, and *CD*, are together, greater than *AB*; *AB*, being equal to *EB*, as radii of the circle *AEG*, the *CD*, and *CB*, together, that is, the *BD*, is greater than *BE*; or, the part greater than the whole, which is impossible; also, *BC*, and *CF*, would be equal to *BC*, and *CA*, and greater than *BG*; but the *CF*=*CD*, and the *BG*=*BE*, as radii of the same respective circles; therefore, also, on the side of *F*, and *G*, the part would become greater than the whole by the equality of these radii; which is impossible.

II. When the circumferences of the circles touch one another externally, the two lines *AB*, and *AC*, are together, greater than the third line *BC*, of the triangle *ABC*; but the *CD*, is equal to *CA*, and the *BA*, equal to *BE*; therefore, also, the *BE*, and *CD*, together, are greater than the *CB*, the part greater than the whole, which is impossible.

Therefore, when two circles touch one another internally or externally, the line joining the centres, (or its protraction,) passes through the point of contact.

§ 44. PROBLEM. **Through a given point to draw a ~~circumference of a~~ circle that shall touch a given circle at a given point.**

APPLIC. *The point* A, *being given, and the circle* BEH, (*fig.* 48,) *the centre of which is* C, *it is required to describe a circle, the circumference of which shall pass through the point* A, *and touch the circumference of the given circle in the point* B; *whether the point* A, *be within or without the circle* BEH.

Solution. Through the centre *C*, of the given circle, and the intended point of contact *B*, draw the indefinite straight line *CBK*, join *BA*, and bisect it in *D;* at the point *D*, erect upon the *AB*, the perpendicluar *DF;* this must cut the *CBK*, in some point like *F*, if *C*, *B*, and *A*, are not in the same straight line. From this point of intersection *F*, and with the radius *BF*, or *AF*, draw the circumference of a circle, it will touch the given circle *EBH*, at the point *B;* as required.

Proof. The line passing through the centres of two circles that touch one another, passing also through their point of contact, the centre of the circle that shall touch the *BEK*, must necessarily lie in the line joining the centre *C*, of the given circle, and the point of contact *B*, or in its protraction; the *AB*, being made a chord of the circle sought, and bisected in *D*, the centre of this circle must (by the Prop. 39,) also lie in the perpendicular *DF*, upon this chord; therefore, this centre must lie in the intersection of the two lines, as in *F;* therefore, the circumference *ABG*, of a circle drawn from *F*, with the radius *FB*, will touch the circle *BEH*, in *B;* as was to be proved.

Corol. 1. If the given point be in one straight line with the centre of the given circle and the point of contact given: the centre of the circle sought lies in that line, or its protraction

and the diameter of it is equal to the distance between the point of contact and the other given point.

Corol. 2. If the radius of the given circle to the point of contact is at right angle to the straight line from the point given without the circle the perpendicular upon the middle of this latter line, will be parallel to the protraction of the radius; they will, therefore, not intersect one another, and no circle can be drawn touching the given circle, the circumference of which will pass through the given point; that is, it becomes undetermined.

§ 45. THEOREM. **The line touching the circumference of a circle, in only one point, is perpendicular to the radius at the point of contact; and inversely: a line perpendicular at the end of the radius of a circle, touches the circumference in no more than one point; that is, it is a tangent to the circle.**

APPLIC. *Let* ADI, (*fig.* 49,) *be a circle, and* C, *its centre, the straight line* GH, *touches the circumference only in the point* A; *then will* GH, *be perpendicular to the radius* CA, *at* A; *and inversely if* GH, *is perpendicular to* CA, *at the point* A; *the* GH, *is a tangent to the circle.*

Demonstr. Suppose the *CA*, be not perpendicular upon *GH*, then any other line as *CB*, will be perpendicular to *GH;* in the triangle *CAB*, the angle *B*, is then a right angle, and as there can be but one right angle in a triangle, and the other angles must be each less than a right angle, the angle *CAB*, must be less than *CBA;* therefore, also, the side *CA*, opposite to this greater angle *B*, must be greater than *CB*, opposite to the angle *A*, as smaller angle; the *CA*, is equal to *CD*, as radius of the same circle; therefore, also, the *CD*, is greater than *CB*, which is drawn from *C*, to a point in the line *GH*, outside of the circle *ADI*, which has only the point *A*, common with it, that is, the part (*CD*,) would be greater than the whole, (*CB*,) which is impossible. Therefore, no straight line can be at right angles upon the *GH*, from the centre *C*, than the radius at the point of contact; therefore, the line touching the circumference of a circle, is perpendicular to the radius at the point of contact.

Inversely. The line *GH*, being perpendicular at the end of the radius *CA*, it will be a tangent to the circle at *A*.

Suppose it was not a tangent to the circle, then it would cut the circumference in one more point, as *I;* join *CI*, then the triangle *CAI*, is isosceles by the two radii *CA*, and *CI*, and the angles at *A*, and *I*, are equal, but the angle at *A*, is a right angle, therefore, also, the angle at *I*, is a right angle, and the triangle *CAI*, has two angles, each equal to one right angle, which is impossible. Therefore, the line *GH*, has not any other point common with the circumference of the circle, except *A*, that is, it is a tangent to the circle. As was to be demonstrated.

Corol. At one point in the circumference of a circle, there can be but one straight line tangent to the circle; every other straight line through that point must cut the circle.

Scholium. If a tangent is to be drawn to the circle from a point in the circumference; the radius to that point being drawn, a perpendicular to it at that point, will be the tangent required.

§ 46. PROBLEM. **To draw a tangent to a circle, from a given point without it.**

APPLIC. *The circle* EDEG, *and the point* A, *without it,* (*fig.* 50,) *being given, it is required to draw from* A, *a tangent to the circle* EDEG.

Solution. Draw *AC*, joining the centre *C*, and the given point *A*, with *CA*, as radius from the centre *C*, draw the circumference of the circle, *ABFB;* at the point *D*, where the *AC*, cut the circumference of the given circle, draw *EDB*, at right angles to *AC*, it will cut the circumference of the outer circle in *B*, and *B;* join the *BC*, which will cut the circumference of the given circle in *E;* join *AE;* and *AE*, will be the tangent to the given circle from the point *A;* as was required.

Proof. In the two triangles *CAE*, and *CBD*, there is *CD=CE*, as radii of the given circle, the *CB=CA*, as radii of the outer circle; the angle at *C*, is common to both triangles

CBD, and *ACE*; therefore, these two triangles are equal, and the angles opposite to the equal sides are equal; therefore, the angle *AEC*, is equal to the angle *BDC*, being opposite to the equal radii of the outer circle; but *BDC*, is a right angle by construction, therefore, also, *AEC*; therefore, the straight line *AE*, is drawn from the point *A*, without the circle, perpendicular to the end of the radius *CE*; therefore this *AE*, is the tangent required.

Scholium. As every straight line drawn within a circle, cuts the circumference in two points, there will always be two points in the circumference of the outer circle above described, from which radii can be drawn to the common centre of both circles, which will determine two points in the circumference of the given circle; these being joined to the given point, will form tangents to the given circle; and no more are possible, because a straight line can cut the circumference of a circle in no more than two points.

§ 47. THEOREM. **In a circle equal chords are equally distant from the centre, smaller chords are at a greater distance, and greater chords at a less distance from the centre; and inversely, chords at equal distances from the centre are equal; chords at a greater distance are smaller; and chords at a less distance are greater; and the diameter is the longest line that can be drawn in the circle.**

APPLIC. *In the circle,* (*fig.* 51,) *let the chords* BA, *and* FE, *be equal, their distances* CH, *and* CI, *from the centre* C, *will be equal. The chord* AD, *less than* AB, *will have its distance* CK, *from the centre greater than the distance* CH, *of the latter. Inversely: the* CH, *and* CI, *being equal, the* AB, *and* FE, *will be equal; the* CK, *being greater than* CH, *the chord* DA, *will be less than* AB. *The diameter* GL, *will be the greatest straight line that can be drawn in the circle.*

Demonstr. The perpendiculars *CH*, *CI*, and *CK*, being drawn upon the several chords, join *CA*, and *CE*; the perpen-

diculars from the centre upon these several chords will bisect them, and the radii of the circle will all be equal; that is, $CA=CE=CG=CL$; then:

I. The chords AB, and EF, being equal, their distances CH, and CI, are equal; for, the CA, being equal to CE, as radii, if from the square of each of them be taken away the square of the equal half chords AH, and CA there remains in the triangles CHA, and CIE, right angled at H, and I, the square of the third side CH, in the one, equal to the square of the third side CI, in the other, the CH, and CI, are the distances of the chords from the centre, and their squares being equal, the lines are equal; therefore, $CH=CI$; that is, the equal chords are equally distant from the centre. Inversely: if from the square of the equal radii CA, and CE, there be taken away the square of CH, and CI, respectively, each from each, the differences will be equal; the CH, and CI, being equal by the supposition of equal distances of the chords from the centres; these differences are the squares of the third sides of the right angled triangles CAH, and CEI, the squares being equal, the sides AH, and EI, are themselves equal; therefore, also, their doubles; that is, $AB=EF$, which are the chords; therefore, at equal distances from the centre, the chords are equal.

II. Let the chord AD, be smaller than AB, then will AD, be farther from the centre C, than AB; that is, the perpendicular CK, from the centre upon AD, will be greater than the perpendicular CH, upon the AB. The radius CA, being common to the two right angled triangles ACH, and ACK, subtracting from the square of AC, as hypathenuse, the square of AH, or of half the chord AB, there remains the square of CH, and subtracting from the same square of CA, the square of AK, there remains the square of CK; but the square of AH, is greater than the square of AK, because the chord AB, is greater than AD, therefore the remaining square CH, is less than the remaining square CK; or CH, is less than CK; that is the distance of the greater chord from the centre is less than

the distance of the smaller chord. Inversely, if from the square of AC, the squares of CH, or of CK, are respectively subtracted, the CK, being larger than the CH, that is, the chord AD, being farther from the centre than the chord AB; thence of the squares remaining by the subtraction in the same right angled triangles as before, the square upon AH, will be greater than the square upon AK, the third sides of these triangles; therefore, also, AH, is greater than AK, therefore also the chord AB, the double of AH, is greater than the chord AD, the double of AK; that is, the chord at the greater distance from the centre is less than the chord nearer to the centre.

III. The diameter is the greatest line that can be drawn in the circle: for $AC=EC=GC=LC$; therefore, the two parts of the diameter CE, and CL, are each equal to the hypothenuse of any of the right angled triangles used above; their sum LG, is therefore greater than any chord in the circle.

All which was to be demonstrated.

§ 48. THEOREM. In the same or in equal circles, equal angles at the centre subtend equal chords, and equal arcs; and inversely, equal chords, or arcs, subtend equal angles at the centre.

APPLIC. *In the same or equal circles* EDAGB, *and* EDF, (*fig.* 52,) *let the angles at the centre* C, *be* ACB= DCE; *then will the chords and the arcs* AB, *and* DE, *be equal; and inversely, if the chords or arcs* AB, *and* DE, *are equal, the angles at the centre* C, *will be equal.*

Demonstr. In the two triangles ABC, and DEC, whether in the same or in equal circles; the sides $AC=BC=DC=EC$; as radii of the same or equal circles, the angles at C, being equal by the supposition; the third sides AB, and DE, are also equal; that is, the chords are equal. Because, every point in the circumference of the same or equal circles, is equally distant from the centre C, the arc AB, being placed upon the arc DE, will coincide with the same, as will also

the whole sector, having the same angle at the centre, therefore, also, these arcs are equal.

Inversely. If the chords, or the arcs, *AB*, and *DE*, are equal, the radii *AC*, *BC*, *DC*, and *EC*, being all equal, the triangles formed by the equal radii, and the equal chords, or arcs, will also be equal, and coincide with one another in all respects; therefore, also, the angle at the centre *C*, will be equal; that is, $ACB=DCE$; as was to be demonstrated.

Corol. The perpendicular from the centre of a circle upon a chord, bisects also the arc subtended by that chord; because it bisects the angle at the centre, and to these equal angles equal arcs will correspond.

§ 49. THEOREM. The angle at the centre of a circle is double the angle at the circumference, that is subtended by the same chord or arc.

APPLIC. *In the circles* ADBE, (*fig.* 53 *and* 54,) *let the angles* ADB, *that has its angular point* D, *in the circumference, be subtended by the same arc or chord as the angle* ACB, *at the centre of the circle, the angle* ACB, *shall be double the angle* ADB. (*This angle is usually said to stand in the segment* ADB.)

Demonstr. Join *CB*, and *DC*, and protract it to the intersection with the circumference at *E*, the angle *ECB*, is outward angle to the triangle *BDC*; therefore, it is equal to the two inward and opposite angles *CBD*, and *CDB*, together; the *CD*, and *CB*, being equal, as radii of the same circle, the angles *CBD*, and *CDB*, are equal; therefore, the angle *ECB*, is double the angle *EDB*; exactly in the same manner is proved, that the angle *ACE*, is double the angle *ADE*; therefore, in the two figures 53, the sum of the two angles *ACE* and *BCE*, that is, *ACB*, is double the sum of the two angles *ADE*, and *BDE*, that is *ADB*; and in figure 54, if from the angle *EDB*, be taken away the angle *EDA*, and from the *ECB*, the *ECA*, the difference between the two last, namely, the angle *ACB*, will be double the angle *ADB*;

therefore, the angle at the centre of the circle is double the angle at the circumference, that is subtended by the same arc or chord; (in all cases whether the centre lies within the angle at the circumference, or without it, and whether it be obtuse or acute.)

Corol. 1. All the angles at the circumference standing in the same segment of a circle are equal, because the above demonstration applies to any point in which the angular point might be placed in the circumference of this segment.

Corol. 2. The angle in a semi-circle is a right angle, because it is half the angle which a diameter cuts off from a circle on one side, or of two right angles, that is the sum of all the angles which a straight line admits on one side.

Corol. 3. The angle *in* a segment *greater* than a semicircle, is *acute:* because it is the half of an angle less than the right angles; and the angle *in* a segment *less* than a semicircle, is *obtuse:* because it is the half of an angle greater than two right angles.

Scholium 1. The two opposite angles of a quadrilateral figure inscribed in a circle, like *ABCD*, (fig. 55,) are together equal to two right angles, because they are together subtended by the whole circumference of the circle, or four right angles, and are at the circumference of the circle; therefore, together equal to the half of this circumference, that is, to two right angles.

Scholium 2. If one of the angles of a quadrilateral figure inscribed in a circle, is a right angle, all four are right angles; because its opposite must also be a right angle; and the chords which their sides form, will be the chords of the half of the angles at the centre, or of right angles, and therefore be all equal; the figure itself will be a regular square inscribed in the circle.

§ 50. THEOREM. The angle made by a tangent and a chord is equal to half the angle at the centre subtended by that chord.

APPLIC. *In the circle* BAE, (*figure* 56,) *the chord* BA, *and the tangent* FAG, *cut one another in the point* A, *where the tangent touches the circumference of the circle, the angle* BAF, *will be equal to half the angle* BCA, *at the centre subtended by the same chord* AB.

Demonstr. From the centre *C*, draw the *CD*, perpendicular to the chord *BA;* this perpendicular *CD*, will bisect the angle *BCA*, at the centre; the *CA*, as radius of the circle to the point of contact of the tangent is at right angle upon the tangent *FG*, therefore the angle *CAF*, is a right angle; the angle *CDA*, is also a right angle, by construction; therefore the sum of the two other angles of the triangle, namely, *DCA*, and DAC, are together equal to one right angle; the sum of the two angles *DAF*, and *DAC*, are also equal to one right angle; from both these sums take away the common angle *DCA*, there remains the angle *BAF*, equal to the angle *DCA;* or the angle between the tangent and the chord, equal to half the angle at the centre of the circle, as was to be demonstrated.

Scholium. The angle at the circumference of the circle being also equal to half the angle at the centre upon the same chord, that is in the alternate segment: the angle between the tangent and the chord is equal to the angle in the alternate segment.

§ 51. PROBLEM. **From a given circle, and from a point given in the circumference of the same, to cut off a segment that will contain a given angle at the circumference.**

APPLIC. *Of the circle* ABD, (*fig.* 57,) *it is required to cut off the segment* BDA, *from the point* A, *that shall contain the given angle* G.

Solution. Draw *EF*, touching the circumference of the circle, at the point *A*, and at *A*, make with *AF*, the angle *FAH*, equal to the given angle *G;* the straight line *AH*, will cut the circumference of the circle in the point *B*, and the segment *ADB*, will contain the angle *G*, at any point of its circumference, as *D*.

Proof. By the foregoing Theorem the angle between the tangent and the chord is equal to the angle in the alternate segment of the circle, therefore the *AB*, being a chord making with the tangent *EF*, an angle equal to the given angle *G*,

by the construction, any angle like *D*, being at the circumference in the alternate segment, is equal to *BAF*, therefore equal to the given angle *G;* that is, the segment *ADB*, will contain the angle required.

§ 52. PROBLEM. **Upon a given straight line to describe a segment of a circle that shall contain a given angle at the circumference.**

APPLIC. *Let* AB, *be the given straight line and* G, *the given rectilineal angle:* (*fig.* 58,) *it is required to describe upon* AB, *a segment* ADB, *of a circle that shall contain at the circumference an angle equal to the given rectilineal angle* G.

Solution. At the point *A*, draw *EAF*, making with the *AB*, the angle *BAF*, equal to the given rectilineal angle *G*, draw *AH*, perpendicular to *EF;* bisect *AB*, in *I*, and from *I*, draw the *IK*, perpendicular to the *AB;* these two perpendiculars will intersect one another, as at *C;* from this point as centre with the radius *CA*, describe the circle *ADB*, the segment *ADB*, will contain at any point of the circumference an angle equal to the given angle *G;* as was required.

Proof. The perpendicular *AH*, upon the tangent to a circle at the point of contact passes through the centre of that circle. The given straight line *AB*, being bisected, the perpendicular *IK*, upon its bisecting point *I*, will pass through the centre of any circle of which *AB*, is considered to be a chord; and the point *C*, the common intersection of these two perpendiculars is the centre of the circle of which the angle *BAD*, made equal to the given rectilineal angle *G*, forms that angle between the tangent and the chord; therefore in a circle drawn with the radius *CA*, the angle in the alternate segment, as *ADB*, will be equal to the angle *BAF;* therefore, also, to the given angle *G*, as has been required.

Corol. A determined straight line will subtend equal angles from any point in the circumference of the circular segment which will contain that angle.

§ 53. PROBLEM. To inscribe a triangle in a given circle that shall be equiangular to a given triangle.

APPLIC. *The triangle* ABC, *and the circle* DGKH, *being given: (fig. 59,) it is required to inscribe in the circle a triangle equiangular to the given triangle.*

Solution. At any point as *D*, in the circumference of the circle *DGKH*, draw *EF*, a tangent to the circle; make the angle *FDG*, equal to the angle *A*, of the given triangle, the *DG*, will cut the circumference of the circle in *G;* at the same point *D*, make the angle *EDH*, equal to the angle *B*, of the given triangle; the *DH*, will cut the circumference of the circle in *H;* join *GH;* the triangle *DGH*, will be the triangle required, equiangular to the given triangle *ABC*.

Proof. The angle *GDF*, being equal to the angle *A*, of the given triangle, by construction, and also to the angle *H*, of the inscribed triangle, because this is the angle in the alternate segment upon *DG*, as chord, the angle *H*, is equal to the angle *A*, of the given triangle; for the same reason, also, the angle at *G*, of the inscribed triangle, is equal to the angle *B*, of the given triangle; the third angles *C*, and *D*, of the two triangles *ABC*, and *HGD*, completing these to two right angles, are therefore also equal, and these two triangles are equiangular; the triangle *DHG*, being inscribed in the circle, (that is, having its angular points in the circumference of the circle,) it is the triangle required equiangular to the given triangle.

Corol. 1. If the triangle is equiangular, then each of the angles is equal to the third part of two right angles, and the triangle forms what is called a regular poligon, inscribed in the circle.

Corol. 2. By the same process as in the problem, evidently, also, any poligon, regular or irregular, that can be inscribed in a circle, can be inscribed in any circle.

Scholium. As the perpendicular upon the chord bisects also the arc, if at the end of this radius, bisecting the arc, tangents be drawn to the circle, these will be parallel to the chords, being both perpendicular to the radius, and the intersections of these tangents will form a triangle or poligon equiangular to the in-

scribed one, the sides of which touching the circumference of the circle, the figure will be circumscribed to the circle.

§ 54. PROBLEM. To circumscribe a triangle, to a given circle that shall be equiangular to a given triangle.

APPLIC. *The given triangle being* ABC, *and the given circle being* GHI, *the centre of it* F; (*fig.* 60,) *it is required to describe a triangle equiangular to* ABC, *the sides of which shall touch the circumference of the circle* GHI.

Solution. Protract the line *AC*, of the given triangle, on both sides, to *D*, and *E;* in the circle draw the radius *FG;* and at the centre *F*, make with *FG*, the angle *GFI*, equal to the outer angle *DCB*, of the given triangle, and the angle *GFH*, equal to the outer angle *EAB*, of the given triangle; through the three points *G*, *H*, and *I*, draw the lines *MK*, *ML*, and *KL*, respectively perpendicular at the end of their corresponding radii *FG*, *FH*, and *FI*, they will be tangent to the given circle, and their intersection will form the triangle *MKL*, equiangular to the given triangle *ABC*, and circumscribed to the given circle.

Proof. The lines *MK*, *KL*, and *LM*, must intersect one another, because they are perpendicular to lines intersecting each other; (as is evident by the formation of the squares and parallelograms;) in the quadrilateral figure *FIMK*, the angles at *I*, and *G*, are right angles, therefore the two other angles *F*, and *K*, must together be equal to two right angles; by construction, the angle *IFG*, is equal to the angle *DCB*, the outer angle in the triangle *ABC*, which makes, with the angle *C*, in the triangle, together, two right angles; therefore, the remaining angle *K*, in the quadrilateral figure *GKIF*, is equal to this angle *C*, of the triangle, and this angle *K*, is also one of the angles of the triangle circumscribed to the circle. Exactly in the same manner, it is proved that the angle *M*, of the triangle circumscribed to the circle, is equal to the angle *A*, of the given triangle; and two angles of these two triangles

being equal, the third angle B, in the one, must also be equal to the third angle L, in the other; therefore, the two triangles ABC, and MKL, are equiangular, and the sides of the MKL, are all tangents to the circle; as was required.

Corol. If the triangle is equiangular, the triangle so described, touching the circle, will be a regular poligon, circumscribed to the circle.

§ 55. PROBLEM. **To inscribe a circle in a given triangle.**

APPLIC. *In the triangle* ABD, (*fig.* 61,) *it is required to inscribe a circle, that shall touch its three sides internally.*

Solution. Bisect the angle BAD, by the line AH, and likewise the angle ABD, by the line BI; from the point C, the intersection of these two lines, draw the perpendiculars CG, CF, and CE, upon the three sides of the triangle, these perpendiculars will be equal; and the circle described from C, as centre, with either one of them, will touch the three sides of the triangle ABD; as was required.

Proof. In the two triangles, CAG, and CAF, the angles at A, are equal, being the halves of the angle A, of the given triangle, the angles at F, and G, are right angles, and the side AC, is common to both triangles; therefore, these two triangles are equal, and the side CG, in the one triangle, opposite to the half of the angle of the triangle, is equal the CF, opposite in the other triangle, to the other half of the same angle. In exactly the same manner, it is proved, that the CE, is equal to CF, by the bisection of the angle B; therefore, the three perpendiculars CG, CF, and CE, are equal, and a circle can be drawn from the centre C, passing through the three points G, F, and E, which will touch the triangle ABD, internally, all the sides being at right angles to their corresponding radii respectively; as was to be proved.

Corol. The three lines bisecting the three angles of a triangle cut one another in one point, which is the centre of the inscribed circle.

§ 56. PROBLEM. An equilateral triangle being given, inscribed in a circle: to find the sides and angles of the regular poligon of six sides, inscribed in the same circle.

APPLIC. *The equilateral triangle* ABC, *inscribed in the circle* AGDEBF, *being given,* (*fig.* 62,) *to find the sides of the six sided regular poligon,* (*that is, the hexadecon,*) *inscribed in the same circle, and the angles formed by the radii and the sides of the hexagon,* C, *being the centre of the circumscribed circle.*

Solution. From the centre *C,* of the circle, draw the perpendiculars upon the sides of the given triangle; protracted to the intersection of the circumference of the circle as *CKF, CIE,* and *CHG;* join the chords *AG, GD, DE, EB, BF,* and *FA*; these will be the sides of the regular figure of six equal sides inscribed in the same circle. These equal sides will be equal to the radius of the circle, and each angle formed by one of these sides and the radius drawn to its angular point, will be equal to one-third of two right angles.

Proof. The perpendiculars from the centre of the circle upon the three sides of the equilateral triangle as chords, divide the arcs *AGD, DEB,* and *BFA,* equally in the points *G, E,* and *F;* therefore, also, all the chords *AG, GD, DE,* and so on, are equal to one another, and subtend equal angles at the centre *C;* and as there are six sides, each angle at *C,* is equal to the sixth part of four right angles, or the third part of two right angles. The figure *AGDEBF,* is a regular hexagon in the circle, and the sum of any equal number of these equal arcs will be equal, and correspond to an equal angle at the circumference; therefore, all the angles, as *AGD, GDE,* and so on, standing in equal segments, are equal to one another; and their sum being equal to twice as many right angles as the figure has sides wanting four: that is, eight right angles, each of these angles of the hexagon is equal eight sixths of one right angle, or two thirds of two right angles, and they are bisected

by the radii, for these radii, as CA, and CG, being equal, the angles opposite to them are equal: that is, $CAG=CGA$, and the sum of these is also equal to two thirds of two right angles, because they make with the angle at C, together, two right angles, the angles at C, are each one third of two right angles; each of the angles of any of the triangles, as CAG, is, therefore, one third of two right angles, and the triangle is equilateral; therefore, also, the sides of the hexagon, as AG, GD, &c. are each equal to the radius of the circumscribed circle; all which was to be proved.

Corol. 1. The radius of the circle will, therefore, cut the circumference in six parts having equal chords and equal arcs.

Corol. 2. By the bisection of the chord of the hexagon, as here done for the triangle, the sides of the regular figure of twelve sides, (or the *dodecagon*,) inscribed in the circle, will be obtained; and so on, for any continued bisection.

Corol. 3. As the square is a regular figure of four sides, and can, therefore, be inscribed in the circle, the same process of bisection will give the side of the regular poligon of eight sides, and any subsequent bisection.

§ 57. PROBLEM. **Two sides and the included angle in a quadrilateral figure that cannot be inscribed in a circle, being given, together with the angle subtended by these lines at the fourth point of the same, to describe the quadrilateral figure.**

APPLIC. *Let* AB, *and* BC, *and the angle* ABC, *which they include, be given,* (*fig.* 63, *a*, *b*,) *let the angle* E, *be the angle subtended at the fourth point by the line* AB, *and the angle* F, *the angle subtended at the same fourth point by the line* BC; *it is required to describe the quadrilateral figure* ABCD, *resulting from it,* (*provided the same cannot be inscribed in a circle.*)

Solution. Upon the side AB, describe a segment of a circle BDA, which will contain at the circumference an angle equal to its corresponding angle E. Upon BC, and towards

the same side, as the arc *BDA*, describe a segment *CDB*, that will contain at the circumference an angle equal to its corresponding angle *F*; join *CD*, and *DA*, the figure *ABCD*, will be the quadrilateral figure required.

Proof. The arc *BDA*, described upon *AB*, as chord of the circle *ABD*, will contain at any of its points in the circumference an angle equal to the given angle *E*, by the construction; in the same manner, the arc *BDC*, described upon *BC*, as chord, will contain in any point of its circumference an angle equal to the given angle *F*, by the similar construction; these two arcs having one point *B*, common, must intersect one another in a second point, if the two circles have not the same centre, as is supposed; this intersection taking place at the point *D*, the two given angles will be subtended simultaneously in that point alone; therefore, the point *D*, where the two arcs intersect one another, is the fourth point of the quadrilateral figure required, which is completed by the lines *DA*, and *DC*; or *ABCD*.

Corol. 1. If the sum of the two given angles is less than two right angles, the fourth point sought lies without the triangle formed by the given sides and the included angle, if they are together equal to two right angles, the fourth point lies in the straight line between the two extreme points; and if the sum of the two given angles is more than two right angles, the fourth point lies within the triangle formed by the three other points, (as in fig. 63, *b*.)

Corol. 2. If the two given lines lie in one straight line, the fourth point forms the third point of a triangle opposite to the sum of the two other sides, as side of the triangle (figure 63, *c*.)

Corol. 3. If the sum of the two angles at the fourth point is less than two right angles, as well as when the two given lines lie in one and the same straight line, there are two fourth points possible, one on each side of the two given straight lines.

Corol. 4. If the centres of the two circles fall together, their circumferences also lie upon one another, and no intersection, therefore, no determination of the fourth point, is obtainable by these data.

(*Note.*) This is generally called the Problem of three points.

§ 58. THEOREM. **Parallel chords in a circle intersect equal arcs of the circumference.**

APPLIC. *The chords* AB, *and* DE, *in the circle* DAFBEG, (*figure* 64,) *being parallel, the arcs* AD, *and* BE, *which they intersect in the circumference of the circle, will be equal.*

Demonstr. Draw the *CF*, perpendicular to the chords, it will bisect both these chords, and the arcs *AFB*, and *DFE*, equally; join *CD*, *CA*, *CB*, and *CE;* the angle *FCA*, will be equal to the angle *FCB*, and the *FCD*, to the *FCE;* subtracting equals from equals, there remain the equal angles *ACD*, and *BCE*, at the centre; and as equal angles at the centre subtend equal arcs, the arcs *AD*, and *BE*, are equal; as was to be demonstrated.

§ 59. THEOREM. **Two chords intersecting one another in a circle, subtend by their vertical angles arcs of the circumference, which are together equal to the two arcs subtended by the diameters parallel to these chords.**

APPLIC. *The two chords* AB, *and* DE, *in the circle* AHDBGEIF, (*figure* 65,) *intersecting one another in* K, *the sum of the arcs* DB, *and* AFIE, *which the vertical angles* DKB, *and* AKE, *subtend, will be equal to the sum of the arcs* FI, *and* HDBG, *subtended by the vertical angles* FCI, *and* HCG, *at the centre* C, *of the circle.*

Demonstr. The diameter *HI*, being parallel to the chord *DE*, the arcs *H*D, and EI, which they intersect at the circumference, are equal; and for the same reason, also, the arcs *AF*, and *BG*, are equal; if to the arc *BD*, be added the two arcs *HD*, and *BG*, the arc *HDBG*, will be the arc subtended by the diameters, under an angle *C*, at the centre equal to the angle *K*, of the intersecting chords; if on the other side from the arc *AFIE*, be taken away the two arcs *AF*, and *EI*, which are equal to the *BG*, and *HD*, the arc *FI*, will be sub-

tended by the angle *FCI*, at the centre, and equal to the other vertical angle *HCG*, and to the two vertical angles *DKB*, and *AKE;* therefore, the sum of the two arcs *DB*, and *AFIE*, is equal to the sum of the two arcs *HDBG*, and *FI*, the two first being the vertical arcs subtended by the chords intersecting one another in *K*, and the two latter the sum of the two vertical arcs subtended by the diameters; as was to be demonstrated.

Corol. The distance between the point of intersection of the chords and the centre of the circle, is called the eccentricity of this point, and half the sum of the vertical arcs of any such point, is equal to the arc subtended by the same angle from the centre.

Scholium. If a series of equal angles be supposed around one point, the sum of which is equal to one whole circumference of a circle, (or four right angles,) and these be placed in any point of the circle without the centre, revolving around that point as centre, the sum of all the differences of the arcs which they will intersect by any two positions, will always be equal to the sum of the differences of the arcs which these would have described by the same revolution made from the centre of the circle; because their sum being in both cases a whole circumference, the sum of the differences, and the angle represented by it at the centre, will be equal.

CHAPTER V.

Of the proportionality of Lines and Figures.

§ 60. PROBLEM. To divide a given terminated straight line into a given number of equal parts.

APPLIC. *The determinate straight line* AB, (*fig.* 66,) *is required to be divided into a certain number of equal parts,* (*suppose in* 5.)

Solution. From *A*, one of the end points of the given straight line draw an indefinite straight line *AC*, making any acute angle with it: upon this *AC*, lay off in succession lines of equal length, in number equal to the number of division required, as *AD=DE=EF=FG=GH;* join *HB*, and through every one of the points, *D*, *E*, *F*, *G*, draw lines parallel to *HB*, as *Da*, *Eb*, *Fc*, *Gd*, cutting the given straight line *AB*, in the points; *a*, *b*, *c*, *d*, and this line will be divided by them into the number of equal parts required, as *Aa=ab =bc=cd=dB;* which was to be done.

Proof. Through the points *D*, *E*, *F*, *G*, draw the *Dn*, *Eo*, *Fp*, and *Gq*, parallel to *AB*, which will intersect the dividing lines *g*, *h*, and *i;* the parallels, *aD*, *bE*, *cF*, *dG*, and *BH*, are cut by the straight line *AH*, they make the inner and outer angles on the same side of it equal, that is, the angles *ADa*, *DEg*, *EFh*, *FGi*, and *GHq*, are equal; in like manner, because the parallels *AB*, *Dn*, *Eo*, *Fp*, and *Gq*, are cut by the *AH*, the angles *DAa*, *EDg*, *FEh*, *GFi*, *HGq*, are all equal; the lines *AD*, *DE*, &c. being all equal by construction, and included between equal angles (as just shewn,) all the triangles *AaD*, *DgE*, *EhF*, and so on, are equal in all respects, therefore also the sides opposite to equal angles are equal, that is *Aa=Dg =Eh=Fi=Gq*. On account of the parallelism of the various lines drawn by the construction, the figures *Db*, *Ec*, *Fd*, and *GB*, are parallelograms, having the opposite sides equal, and

as all these sides are equal to Aa, there will also be $Aa=ab=bc=cd=dB$, that is the line AB, has been divided into the number of parts required, all equal to one another; which was to be proved.

Corol. All the individual triangles formed by the foregoing construction, by any of the parallel lines with its opposite angular point, are equiangular among themselves and with the whole triangle formed by ABH, formed by the extreme lines, (that is they are all similar triangles.)

Scholium. It might already from the result of this Problem be deduced that triangles are to one another in the ratio of the squares of their sides opposite to equal angles, or what is called homologous sides. For every parallelogram that resulted being discomponible in as many triangles equal to the AaD, there will be found in each of the triangles made by any of the parallel lines as many such equal triangles as the square of the side indicates, taking the side of the triangle AaD, as unity; as for instance AcF, having three parts $Aa+ab+bc$, will contain nine triangles equal to AaD; and so for any other of the triangles presented by the figure.

§ 61. PROBLEM. To find the greatest common measure between two given straight lines.

APPLIC. *The two straight lines* AB, *and* CD, *being given: (fig.* 67,) *it is required to find their greatest common measure, if they have any.*

Solution. From the greater line AB, take away the smaller CD, as often as it may be taken, in the figure: $Aa=ab=bc=CD$; leaving the remainder cB, smaller than CD; from the smaller line CD, take away the remainder cB, as often as it may be taken, as $Cd=dg=cB$, and leaving gD, as a remainder, smaller than cB; then take away from the former remainder cB, this second remainder gD, as often as it may be taken. Continue this process until the remainder last obtained measures the preceding one exactly; that is, contains it a full number of times. Suppose in this case the gD, to have been contained exactly twice in cB; that is, that it would have given

$gD=cf=fB$, then the line gD, will be the greatest common measure between AB, and CD; which was required.

Proof. The greatest line that can be a common measure between two given lines, can be no greater than the smaller line itself, as here CD, this being taken away from the greater line AB, as often as it is contained in it whole times, and leaving cB, as remainder, smaller than CD; any line that shall measure the two given lines without remainder must measure the cB, and the CD, which together measure the AB; between these two lines cB, and CD, there can again be no greater common measure than cB, itself; this being taken away as often as admissible, and leaving the remainder, gD, the same reasoning holds good between the gD, and cB, as between the former remainder and the smaller line CD, and before between CD, and AB; hence gD, being found to measure the cB, without remainder, this gD, will also measure the Cg, a multiple of cB; thence it will also measure the cD, therefore also the Ac, a multiple of CD, and as it measures also the remainder cB, it measures the whole AB; therefore, the gD, is a common measure between the two given lines AB, and CD; and it is the greatest possible, because of all the lines successively occuring there can never be a greater common measure than the smaller, which were always the successive remainders; if any common measure is possible it must be one of the remainders successively obtained by this process, and if none is found by this method there is no common measure between them.

§ 62. THEOREM. **Triangles that are between the same parallels, (or that are of the same altitude,) are to one another in the same ratio as their bases.**

APPLIC. *If the triangles* ABC, *and* DEF, (*fig.* 68,) *have the same altitude, that is: that they are, or can be, between the same parallels; these triangles will have to one another the same ratio as their bases, whether these bases are commensurable or not; that is, there will be:*

ABC : DEF=AB : DE.

Demonstr. 1. When the two bases have *a common measure.* Suppose this to be the Aa, and therefore the AB, and the DE, be both divided into equal parts $Aa=ab=bc=cB=Dd=de=eE$; join aC, bC, cC, in the triangle ABC; and dF, eF, in the triangle DEF, the resulting triangles will all be equal to one another, as $AaC=abC=bcC=cBC=DdF=deF=eEF$; and the whole triangles ABC, and DEF, will be to one another as the number of these equal triangles which they contain; the number of these triangles in each being equal to the number of these equal bases, these triangles are to one another as their bases, or the proportion results;

$$ABC : DEF = AB : DE;$$

which was to be demonstrated.

2. When the two bases have *no common measure.* Suppose AB, and DE, (in figure 68*b*,) have no common measure, the triangles ABC, and DEF, are to one another as their bases AB, and DE; suppose they were not to one another in that ratio, but that they were to another as AB, to Df, smaller (or greater) than DE; find a subdivision of AB, so that the parts be smaller than fE, and suppose them laid off in DE, until one of the subdivisions fall between f, and E, in the point g; which must always take place, because the subdivisions have been taken smaller than fE; join gF, and fF; then by preceding part of the demonstration there will be

$ABC : DgF = AB : Dg$; but, by the supposition $ABC : DEF = AB : Df$;

the triangle DgF, must be smaller than DEF; therefore, also, the Dg, must be smaller than the Df, which is contradictory, as the point g, must fall between f, and E, and in the manner inversely, if the Df, was supposed greater than DE; the Df, would be shewn smaller than Dg, while by the supposition it would be larger; therefore, as the two triangles cannot be to one another either as bases smaller or greater than AB, to DE, they must be to one another as these bases, when their bases have no common measure.

Corol. Parallelograms upon equal bases, and of the same altitude with triangles, being the double of these triangles, the paral-

lelograms of equal altitudes, are also to one another as their bases.

§ 63. THEOREM. **If two sides of a triangle, or their protractions, are cut by a line parallel to the third side, these lines will be cut proportionally: that is, their parts intersected will have the same ratio to one another; and inversely, if two sides of a triangle are cut proportionally, the cutting line will be parallel to the third line.**

APPLIC. *Let the sides* AB, AC, (*fig.* 69,) *or their protraction on either side of the angular point* A, *be cut by the line* DE, *parallel to* BC; *the lines* AB, *and* AC, *or their protraction, will be cut proportionally; that is: there will be* AB : BD=AE : EC; *and all the consequent mutations that follow from the principles of proportions.*

Demonstr. Join *BE*, and *CD;* the two triangles *BCD*, and *BCE*, are equal, having the same base *BC*, and being between the same parallels *BC*, and *DE;* the two triangles *ABC*, and *BCD*, having the point *C*, common, and their bases in the same straight line, passing through *A*, *D*, and *B;* they are to one another as their bases *AB*, and *BD;* that is, they give the proportion

$$ABC : BCD = AB : BD.$$

Exactly in the same manner, the two triangles *ABC*, and *CBE*, having the point *B*, common, give the proportion

$$ABC : CBE = AC : CE,$$

because the two triangles *BCD*, and *CBE*, are equal, the two first terms in these two proportions being equal, the two second terms are equal also; that furnishes the proportion

$$AB : BD = AC : CE,$$

which applies equally to all the three figures; and furnishes all the mutations of the proportions that are resulting from them by the principles of the proportions; as was to be demonstrated.

Inversely. When the sides *AB*, and *AC*, of a triangle, or their protraction on either side of *A*, are cut proportionally by *DE*, the *DE*, will be parallel to *BC*, the third line of the triangle *ABC*. Suppose *DE*, not parallel to *BC;* then

through D, draw the DF, parallel to BC; this will by the preceding give the proportion

$AB : BD = AC : CF$; but, by the supposed proportionality,

$AB : BD = AC : CE$; therefore, also,

$AC : CF = AC : CE$; which is impossible,

CF, being supposed different from CE; the same would be the case with any other point like E; therefore, the line cutting two sides of a triangle proportionally, is parallel to the third side of the triangle; as was to be demonstrated.

Corol. Also, the perpendicular from an angular point of a triangle upon the opposite side, will be cut by a line parallel to this third line in the same proportion as these lines.

Scholium 1. Any line cutting the two sides of a triangle parallel with the third side, will make with these sides angles which are equal to the angles which this third side makes with them; the resulting triangles will, therefore, be equiangular; triangles having this property, are called *similar triangles.*

Scholium 2. By the principles of this Theorem, a determinate straight line can be divided proportionally to the given parts of another straight line, by placing these two lines, so as to form an acute angle together, joining the end points of the lines, and through the other points of sub-division, drawing lines parallel to this extreme line.

§ 64. THEOREM. Similar triangles have their sides about the equal angles proportional; and inversely: if two triangles have one angle equal, and the sides about this equal angle proportional, they will be similar; that is, equiangular to one another.

APPLIC. *Let the two triangles* ABC, *and* DEF, (*fig.* 70,) *be similar or equiangular to one another; the sides about the equal angles* A, *and* F, *are proportional; and inversely: if the sides about these equal angles are proportional; that is, if:* AB : AC=FD : FE; *then the triangles will be similar, that is, equiangular; the angles* B, *and* C, *will be equal to the angles* D, *and* E, *respectively.*

Demonstr. When the triangles are similar. From the angular point A, cut off from AB, the one side of the triangle ABC, the $AG=FD$; from AC, the other side, including the equal angle at A, cut off the AH, equal to the other side FE, including the angle F, which is equal to A; join GH. Because the angle A, is equal to the angle F, and the sides AG, and AH, equal to the two sides FD, and FE, respectively, the triangles AGH, and FDE, are equal, and the angles at D, and G, are equal; as also, the angles E, and H; but, these angles are also equal, respectively, to the angles B, and C, of the triangle ABC, by the condition of the similarity of these triangles; therefore, the line GH, is parallel to BC; which furnishes, by the foregoing Proposition, the proportion:

$$AG : AH = AB : AC;$$

therefore, also, by equality of the triangles, AGH, and FDE,

$$FD : FE = AB : AC.$$

Inversely. If the lines AC, and AB, in the triangle ABC, that include the angle A, have the same ratio as the sides FD, and FE, of the triangle FDE, that lie about the angle F, equal to A, then the triangles ABC, and FDE, will be similar: that is, equiangular. For if about the angle A, equal to F, lines AG, and AH, be taken, equal to FD, and FE, these lines will form the triangle AGH, having two sides and the included angle equal, which therefore are equal to one another in all respects; and as the proportionality of the sides about the equal angles gives: $AB : AC = FD : FE$; therefore also by equality, $AB : AC = AG : AH$;

the two lines BC, and GH, are parallel, and the angles which they make with either AB, or AC, are equal; that is, these triangles are similar; the triangles AGH, and FDE, having two sides and the included angle equal are equal in all respects; therefore also the triangle FDE, is similar, that is equiangular to the triangle ABC. As was to be demonstrated.

Corol. 1. This is generally expressed: in similar triangles the sides opposite to equal angles, are homologous, and inversely, the angles opposite to the homologous sides, are equal.

Corol. 2. Similar parallelograms have their sides about the equal angles, proportional; or inversely: when parallelograms having one angle equal, have the sides about this angle proportional, they are similar; because, every parallelogram can be divided into two equal triangles, each having two sides common with the corresponding parallelograms.

§ 65. THEOREM. **If one angle of a triangle be bisected by a line cutting the opposite side, the segments which it makes of this opposite side, will be proportional to the sides of the triangle adjacent to them.**

APPLIC. *Let* ABC, (*fig.* 71,) *be the triangle in which the angle* C, *is bisected, and this bisecting line cuts the* AB, *in* D; *the segments* AD, *and* BD, *will have the same ratio as the sides* AC, *and* BC, *have: that is, there will be;*

$$AC : CB = AD : DB.$$

Demonstr. Protract AC, and through B, draw a line parallel to CD, intersecting the protracted AC, in the point E, the angles ACD, and AEB, will be equal, because the AE, cutting the two parallel lines CD, and EB, makes inner and outer angle on the same side equal; because the same parallel lines are cut by the CB, the alternate inner angles are equal, that is $DCB = CBE$; the two angles ACD, and BCD, being equal by the bisection of ACB, the angles CEB, and CBE, will be equal, because they are each equal to the half of the same angle, therefore the sides CB, and CE, are equal also; because the CD, and EB, are parallel and cut the sides AB, and AE, about the angle A, these sides are cut proportional: that furnishes the proportion;

$$AC : CE = AD : DB; \text{ and because } CE = CB,$$
$$AC : CB = AD : DB,$$

that is: the segments of AB, made by the CD, bisecting the opposite angle are as the sides of the triangle adjacent to them; which was to be demonstrated.

§ 66. THEOREM. **When in a right angled triangle a perpendicular is drawn from the right angle upon the hypothenuse, this perpendicular is a mean propor-**

tional between the two segments of the hypothenuse; and each side about the right angle is a mean proportional between the hypothenuse, and the segment of it adjacent to that side.

APPLIC. *Let the triangle* ABC, (*fig.* 72,) *be right angled at* A, *and the perpendicular,* AD, *be drawn from the right angle upon the hypothenuse* BC, *then will the following proportions take place as enounced, viz:*

BD : AD=AD : DC.
BC : AB=AB : BD
BC : CA=CA : CD.

Demonstr. Because the *AD*, is perpendicular upon *BC*, the triangle *ABC*, is divided by it into two right angled triangles; the triangle *ABC*, has with the triangle *ABD*, the angle *B*, common, and with the triangle *ACD*, the angle at *C*; the angles at *D*, being right angles as also the angle *A*, of the triangle *ABC*, these three triangles are all equiangular; therefore, their sides opposite to the equal angles, or their homologous sides, are proportional; which gives the proportions following:

by the triangles	*ADB*, and *ADC*;	*BD* : *AD*=*AD* : *CD*
	ADB, and *ABC*;	*BC* : *AB*=*AB* : *BD*
	ABC, and *ADC*;	*BC* : *AC*=*AC* : *DC*;

which was to be demonstrated.

Scholium 1. By this Theorem, a mean proportional between two given lines is found by laying them in continuance of each other in one straight line, describing a semicircle upon their sum, (as in the figure *CD*, and *DB*,) erecting a perpendicular upon the dividing point (*D*;) this perpendicular will be intersected by the semicircle; in a point like *A*, for the angle in the semicircle is always a right angle.

Scholium 2. The (content of the) rectangular parallelogram being equal to the product of its two sides into one another, in a square these two sides being equal: the above principle evidently furnishes the means to make a square equal to a given rectangular parallelogram; and as every oblique angular parallelogram is equal

to a rectangular one of equal base and altitude, any parallelogram can, by the above Theorem, be reduced to an equilateral, (or a square,) and by the proportionality of triangles to parallelograms, also, a square can be constructed equivalent to a given triangle, or to any sum of them.

Scholium 3. The side of a square equal to the difference of two given squares, may also be determined by this Theorem: by describing a semicircle upon the side of the larger of the squares, and with the side of the other intersecting the semi-circumference, the chord of the remaining arc will be the side of the square equal to the difference of the two given squares.

Scholium 4. From this Theorem, united to the property of the semicircle to contain always a right angle at the circumference, is deduced the principle of the determination of all the points of the circumference of a circle by a distance taken upon its diameter, and the length of the perpendicular corresponding to that distance; that is, the *equation of the circle* upon its diameter; which, therefore, gives always the perpendicular a mean proportional between the two parts of the diameter.

§ 67. PROBLEM. To find a third proportional to two given straight lines.

APPLIC. *Let it be proposed to find a third proportional to the two straight lines,* x, *and* y; (*fig.* 73.)

Solution. Draw AF, and AG, cutting one another, under any angle; from the point of their intersection, A, lay off in AF, the $AB=x$; and from the same point on the AG, the $AC=y$. Upon AF, lay off again $AD=y$; join BC, and through the point D, draw the DE, parallel to BC, intersecting AG, in E; then the straight line AE, will be the required third proportional to the two given lines.

Proof. The sides AB, and AC, in the triangle ABC, protracted, being cut by the straight line DE, parallel to the third side BC, the proportions resulting by the foregoing proposition are, $AB : AC = AD : AE$; and because of $AB=x$, and $AD=y=AC$, by the construction this proportion becomes

$$x : y = y : AE;$$

which is, therefore, the third proportional required to be found.

Scholium. If, therefore, one of the sides of a parallelogram be given, together with its content, the perpendicular, or altitude, of the same will be determined by the above proposition; and this rectangular parallelogram may be mutated in one under any angle whatsoever having the same content.

§ 68. PROBLEM. To find a fourth proportional to three given lines.

APPLIC. *The three lines* x, y, *and* z, *being given to find a fourth proportional to them; (fig. 74.)*

Solution. Draw two indefinite straight lines AG, and AF, cutting one another in A, under any angle; on the AG, take $AB=x$; on AF, take $AC=y$; and again on AB, take $AD=z$; join BC, and through D, draw the line DE, parallel to BC, cutting the AF, in the point E, the line AE, will be the fourth proportional to x, y, and z; as required.

Proof. The line DE, cuts the protracted sides AB, and AC, of the triangle ABC, parallel to the third side BC; therefrom is obtained the following proportions:

$$AB : AC = AD : AE;$$

and because $AB=x$; $AC=y$; and $AD=z$; also,

$$x : y = z : AE;$$

therefore, the AE, is the fourth proportional to the three given lines x, y, and z; which was to be found.

Scholium 1. As by the principles of proportions the products of the extreme terms, is equal to the product of the mean terms; and by Geometry the products of the base of a parallelogram into its altitude is equal to its content; it is evident, by the above, that: a content of a parallelogram being given, together with the base of another one, that shall be made equal to it, the fourth proportional found above will give the altitude of the parallelogram required; or, if the altitude was the given part, the fourth proportional found, would be the base of the same.

Scholium 2. The bases and the altitudes of two equal parallelograms, are reciprocally proportional.

Scholium 3. All the consequences of the two preceding Scholiums, apply equally to triangles as halves of the parallelograms.

§ 69. THEOREM. **Similar triangles are to one another in the ratio of the squares of their homologous sides.**

APPLIC. *The triangles* ABC, *and* DEC, *being similar, having the angles at* C, *equal, and the angle at* A, *equal to the angle at* D, (*fig.* 75,) *and therefore,* BC : CE=AC : CD, *then they will be to one another in the ratio of the squares of their sides opposite to equal angles; that is, there will be:* ABC : DCE=AC^2 : CD^2 =BC^2 : CE^2.

Demonstr. The two similar triangles being joined by their equal angles at *C*, so that the sides about the same angle lie in the same straight line, as *BCE*, and *ACD*; join *AE*; the two triangles *ACB*, and *ACE*, having the point *A*, common, have the same altitude; they are, therefore, to one another as their bases *BC*, and *CE*; for the same reason, the triangle *ACE*, is to the triangle *CED*, as the side *AC*, to the side *CD*; thence arise the proportions

$$ABC : ACE = BC : CE,$$

and

$$ACE : CED = AC : CD.$$

By the composition of these two proportions is obtained

$$ABC : CED = BC \times AC : CE \times CD.$$

The proportionality of the sides of the triangles gives

$$BC : CE = AC : CD;$$ therefrom follows

$$ABC : CED = BC^2 : CE^2 = AC^2 : CD^2;$$

and because the same would apply if the triangles were joined by one of the other equal angles, this applies equally to the squares of the third lines AB^2, and DE^2; which was to be demonstrated.

Corol. 1. The first result of the compound proportion shews: that it is the same as the ratio compound of two sides of the similar triangles.

Corol. 2. Parallelograms being double the triangles of equal base and altitude, the same principle of proportionality of the squares of the homologous sides applies equally to similar parallelograms.

Corol. 3. As every rectilineal figure can be considered as compounded of triangles: similar rectilineal figures will be decomposable in similar triangles; they are, therefore, also, in the same ratio as any of these triangles: that is, similar rectilineal figures, in general, will be in the ratio of the squares of their homologous sides.

§ 70. PROBLEM. **To construct a rectilineal figure equal to a given square, and similar to another rectilineal figure.**

APPLIC. *The rectilineal figure* ABCDE, *being given, (fig.* 76,) *it is required to construct a rectilineal figure similar to it, but equal in content to a given square* M.

Solution. Make the square N, equal to the given rectilineal figure $ABCDE$, make an angle G, equal to the angle A, of the given figure; and determine the lines GH, and GL, corresponding (or homologous,) to the AB, and AE, in the given rectilineal figure, and including the equal angles, by the proportion of which the first ratio is that of the sides of the two squares; by that will be obtained:

side of N : side of $M = AB : GH = AE : GL$;

constructing, then, the angles GHI, and GLK, equal respectively, to the angles ABC, and AED, of the given rectilineal figure, determine the lines HI, and LK, which are adjacent to these equal angles, by the same proportion as the GH, and GL, have been determined; thus continue for any number of lines or angles that the given rectilineal figure may present; the resulting rectilineal figure will be *equal* to the given *square*, and *similar* to the *rectilineal figure* given.

Proof. The square N, being equal to the given rectilineal figure, and the square M, being given to make the other rectilineal figure equal to it, the ratio of these two squares will be the ratio of the rectilineal figures, (by § 69,) and the sides of the rectilineal figure required, must be to those of the given rectilineal figure, as the side of the given square to the side of the square equal to the given rectilineal figure; that is, as the side

of *N*, to the side of *M*. The figure *GHIKL*, being therefore made equiangular to the *ABCDE*, and the sides about the equal angles, in the ratio of the sides of the squares *M*, and *N*, the rectilineal figures will be in the ratio of the squares *M*, and *N*; that is, the rectilineal figure *GHIKL*, will be equal to the square *M*. Or there will be,

$$ABCDE : GHIKL = N : M,$$

and because, $N = ABCDE$, by construction;

there is obtained by the above construction,

$$GHIKL = M.$$

As was required.

Corol. 1. If it is required to make a rectilineal figure equal to another rectilineal figure, without its being similar to it, it becomes necessary first to determine the square to which it is equal.

Corol. 2. By the process analogous to this proposition, also figures may be constructed, equal to the sum or difference of others, and similar or equiangular to any given figure.

§ 71. THEOREM. **When two straight lines that intersect one another, either within or without a circle, cut also the circumference of it in two points, they will be cut reciprocally proportional; and if from the point of intersection outside of the circle, a tangent be drawn to the circle, this tangent will be a mean proportional between the two distances of that point from the two intersections of a straight line with the circumference of the circle.**

APPLIC. *The two lines* AD, *and* BE, *or their protraction, intersecting one another in the point* C, *either within or without the circle* ABEDF, (*fig.* 77,) *cutting the circumference of the circle in* A, B, D, *and* E, *the distances, or sections of these lines* AC, CD, *and* CB, CE, *will be reciprocally proportional; and if from* C, *without the circle, the* CF, *be drawn, touching the circumference in* F, *the* CF, *will be a mean proportional between* CD, *and* CA, *or* CE, *and* CB.

Demonstr. 1. Join BD, and AE; in the two triangles ACE, and BCD, the angle DBE, and DAE, are equal as angles at the circumference of a circle, standing in the same segment; for the same reason, also, the angles at D, and E, are equal; the angles at C, are equal, as vertical angles within, and as identical without, the circle; therefore, these triangles are similar, and their homologous sides are proportional, or the following proportion results, viz:

$$CA : CE = CB : CD;$$

or, $$CA : CB = CE : CD;$$

But these lines are the segments reciprocally of the two sections of the lines crossing each other in C, between the point of intersection and the circumference of the circle, therefore these lines are cut reciprocally proportional; as was to be demonstrated.

II. From the point F, where the tangent from C touches the circumference of the circle draw FB, and FE; the angle CFE, between the tangent CF, and the chord FE, is equal to the angle FBE, in the alternate segment; the angle at C, is common to the two triangles CFE, and CFB, therefore these triangles are similar, and their homologous sides are proportional: that is,

$CE : CF = CF : CB$; and by the foregoing proportion also $CD : CF = CF : CA$;

that is: the tangent CF, is a mean proportional between the two segments of a line drawn from the same point C, without the circle, that cuts the circumference in two points.

§ 72. **Problem.** **To divide a given straight line in extreme and mean ratio; that is: so that the whole line shall be to the greater part, as this greater part to the smaller part.**

Applic. *The given straight line* AB, (*fig.* 78,) *shall be divided in extreme and mean ratio; that is: so that the following proportion result, viz:*

AB : AF = AF : FB.

Solution. At the point B, erect a perpendicular upon AB, and make BC, equal to the half of AB; with BC, as radius, from C, as centre, describe the circle EBD; join AC, and protract it to the intersection of the circumference in E; the AE, will be cut by this circumference in the two points D, and E; from AB, cut off the AF, equal to AD; the point F, will divide the given straight line AB, in extreme and mean ratio; that is, it will give the proportion:

$$AB : AF = AF : FB.$$

Proof. By the foregoing Theorem, second part, was demonstrated that AB, as tangent from A, to the circumference of the circle at B, is a mean proportional between the two segments AE, and AD; thence here

$$AE : AB = AB : AD;$$

and by the differences according to the principles of proportions,

$$AE - AB : AB = AB - AD : AD;$$

and because $AB = 2CB = DE$, and also $AD = AF$, and $AB - AD = AB - AF = FB$, by the construction; this proportion changes thus:

$$AD : AB = AB - AF : AF \quad \text{that is}$$
$$AF : AB = FB : AF \quad \text{or, which is the same thing}$$
$$AB : AF = AF : FB;$$

that is: the AB, is divided in the point F, so as to make AF, a mean proportional to the whole AB, and the other part of it FB, as has been required.

§ 78. PROBLEM. To determine the side of a regular decagon or a rectilineal figure of ten sides inscribed in the circle; and also the side of the regular pentagon, or of five equal sides.

APPLIC. *In the circle* AFDBG, (*fig.* 79,) *the centre of which is* C, *it is required to describe the side of the regular decagon, and of the regular pentagon.*

Solution. Divide the radius CB, of the circle in extreme and mean ratio in the point E, the greater part of it; with CE, lay off the chord BD, it will be the side of the regular deca-

gon inscribed in the circle; and the double of the arc which it intersects, or BDF, will have for its chord the side of the regular pentagon.

Proof. The radius CB, being divided in E, so as to make the proportion $CB : CE = CE : EB$; and by construction $BD = CE$; join CD, and ED; the two triangles CBD, and BDE, have the angle at B, common; and because of $BD = CE$, they have the sides about these angles proportional, that is:

$$CB : BD = BD : EB,$$

therefore, these triangles are similar, and the angle EDB, is equal to the angle C, at the centre. The triangle BCD, is isosceles, therefore, also, the triangle EBD; the angles BED, EBD, and CDB, are equal, and the side ED, is equal to BD, thence also to CE; therefore, also, the triangle CED, is isosceles, and the angle $ECD = CDE$. The angle BED, is outward angle to the triangle CED, therefore, equal to the sum of the two inward and opposite angles ECD, and CDE, together, therefore, it is double the angle C, at the centre; and because of $BED = CBD = CDB$, also, the angle CBD, or CDB, is double the angle C, at the centre; therefore, the angle C, is one-fifth part of two right angles; the chord BD, therefore, subtends an angle at the centre equal to the fifth part of the semi-circumference; that is, the tenth part of the whole circumference; it is, therefore, the chord of the regular decagon inscribed in the circle. Laying of the chord BD, from D, to F; the chord BF, will be the side of the regular pentagon, because the arc BF, is double the arc BD, and therefore contained only five times in the whole circumference.

Corol. The chord of the pentagon can evidently also be determined by drawing from A, the other end of the diameter, the AF, parallel to CD; the arc intercepted will again be the BF, double of BD, because it subtends at the circumference an angle equal to half the angle at the centre.

Scholium. The difference between the arc of the decagon, and that of the hexagon, (determined by § 56:) is the arc of the regular poligon of fifteen sides, inscribed in the same circle:

for the arc of the decagon being $\frac{1}{10}$ and that of the hexagon $\frac{1}{6}$ of the circumference, their difference $\frac{1}{6}-\frac{1}{10}=\frac{4}{60}=\frac{1}{15}$ is therefore the arc of the poligon of 15 equal sides. In like manner, the difference between the arc of the pentagon and that of the hexagon, is equal to the arc of the poligon of 30 equal sides; that is: their difference gives $\frac{1}{5}-\frac{1}{6}=\frac{1}{30}$ or an arc that will be contained 30 times in the circumference of the circle. The difference of the arcs of the square and the hexagon, is the arc of the regular dodecagon for the difference: $\frac{1}{4}-\frac{1}{6}=\frac{2}{24}=\frac{1}{12}$; the difference between the arc of the square and that of the pentagon will be, $\frac{1}{4}-\frac{1}{5}=\frac{1}{20}$ or the arc of the regular poligon of 20 sides.

CHAPTER VI.

Application of Arithmetic to Plane Geometry.

§ 74. PROBLEM. **To determine one of the sides of a triangle by the two others, and the segment of the one of them made by a perpendicular from its opposite angle.**

APPLIC. *The side* AC, *in the triangle* ABC, (*fig.* 80,) *shall be determined by the sides* AB, BC, *and the segment* BD, *made by the perpendicular* CD, *from the angle* C, *opposite to* AB, *whether the triangle be acute or obtuse angled.*

Determination I. If the triangle is acute angled, then by the § 36, the perpendicular CD, will be determined equally by the two following expressions, or what might be called equations, as in Algebra, as they are to be used as such; viz:

$$CD^2 = CA^2 - AD^2 = CB^2 - DB^2;$$

and because $AD = AB - BD$; the equation becomes

$$CA^2 - (AB - BD)^2 = CB^2 - DB^2;$$

or $CA^2 - AB^2 - BD^2 + 2AB \times BD = CB^2 - DB^2$;

that is, $CA^2 - AB^2 + 2AB \times BD = CB^2$;

thence $CA^2 = CB^2 + AB^2 - 2AB \times BD$;

which was required to determine.

II. If the triangle is obtuse angled, the simple consideration of CD, falling outside of the triangle, thence being added to AB, instead of subtracted, as above, to give the value of AD, or that $AD = AB + BD$, as by the figure, will change the sign in the term in which BD, is a factor, and therefore give the result immediately.

$$CA^2 = CB^2 + AB^2 + 2AB \times BD;$$

which was required.

§ 75. PROBLEM. The three sides of a triangle being given to determine the perpendicular upon one of the sides from the opposite angle; and the content of the triangle.

APPLIC. *In the triangle* ABC, (*fig.* 80,) *having given the three sides*, AB, AC, *and* CB, *to determine the perpendicular* CD, *and the content of the triangle.*

Determination. By the foregoing proposition (in the first case) has been determined

$$AC^2 = AB^2 + CB^2 - 2AB \times BD$$

from which follows $2AB \times BD = AB^2 + CB^2 - AC^2$

and dividing by $2AB$

$$BD = \frac{AB^2 + CB^2 - AC^2}{2AB}$$

by § 36, there is also

$$CD^2 = CB^2 - BD^2$$

thence if the above value of BD^2, is substituted:

$$CD^2 = CB^2 - \left(\frac{AB^2 + CB^2 - AC^2}{2AB}\right)^2$$

reducing the right hand parts to a common denominator, ob-

serving that this denominator is to be squared; there results:

$$CD^2 = \frac{4CB^2 \times AB^2 - (AB^2 + CB^2 - AC^2)^2}{4AB^2}$$

The two parts of the numerator are the difference of two squares: therefore they are equal to the product of sum and difference of the roots, therefrom results:

$$CD^2 = \frac{(2CB \times AB + AB^2 + CB^2 - AC^2)(2CB \times AB - AB^2 - CB^2 + AC^2)}{4AB^2}$$

and on account of the full squares presented in each of the parts inclosed in the parenthesis, this expression becomes

$$CD^2 = \frac{((AB + CB)^2 - AC^2)(AC^2 - (AB - CB)^2)}{4AB^2}$$

Here the parts in each parenthesis present again the difference of two squares, which will be equal to the product of sum and difference of the roots, by the insertion of which the equation becomes:

$$CD^2 = \frac{(AB + CB + AC)(AB + CB - AC)(AC + AB - CB)(AC + CB - AB)}{4AB^2}$$

Whence by the extraction of the square root the perpendicular is obtained:

$$CD = \frac{1}{2AB}\left[(AB + CB + AC)(AB + CB - AC)(AC + AB - CB)(AC + CB - AB)\right]^{\frac{1}{2}}$$

If the half sum of the three sides of the triangle be expressed in one single letter, the expression will simplify as follows: making what is called the half perimeter, that is:

$$P = \frac{AB + CB + AC;}{2}$$

the following values will result for the four different factors contained under the radical, viz:

for the first; $2P = AB + BC + AC;$

for the second; $P - AC = \frac{AB + BC + AC}{2} - AC;$

$$= \frac{AB + BC + AC - 2AC}{2}$$

$$= \frac{AB + BC - AC;}{2}$$

therefore $2(P-AC) = AB + BC - AC;$

and for the other two factors will be obtained the similar expressions:

$$2(P-BC) = AB + AC - BC,$$

and $$2(P-AB) = AC + BC - AB.$$

These values being substituted for the factors in the expression for the perpendicular, this expression becomes:

$$CD = \frac{1}{2AB}\left(16P(P-AC)(P-BC)(P-AB)\right)^{\frac{1}{2}}$$

The square root of 16, being 4, if this is put outside of the radical expression, and the two in the denominator of that part be compensated for a two as factor of 4, in the numerator, the result becomes as follows:

$$CD = \frac{4}{2AB}\left(P(P-AC)(P-BC)(P-AB)\right)^{\frac{1}{2}}$$

$$= \frac{2}{AB}\left(P(P-AC)(P-BC)(P-AB)\right)^{\frac{1}{2}}$$

which is the determination required, reduced to its most convenient expression.

II. To determine the content of a triangle: the perpendicular, as has just been determined, is to be multiplied by the side upon which it falls, and the product divided by two.

The above expression being that of the perpendicular upon the side AB, the content of the triangle will be the product of the above expression by $\frac{AB}{2}$: or calling $S =$ content of the triangle; and compensating $\frac{2}{AB}$ and $\frac{AB}{2}$

$$S = \frac{CD \times AB}{2}$$

$$= \left(P(P-AC)\,(P-BC)(P-AB)\right)^{\frac{1}{2}}$$

which is again the most convenient expression for the content of the triangle, or the value required to be determined.

If the triangle is obtuse angled the result is identically the same, because the change of sign of the segment of the base

that has been seen in the foregoing proposition, compensates by the alternating of the mutations of the subtractions in the 3d and 4th term under the radical.

Thence in any case the content of a triangle is equal to *the square root of the continued product of the half perimeter of the triangle into the differences between this half perimeter and every one of the sides alternately.*

§ 76. PROBLEM. **To determine the diameter of a circle circumscribed to a given triangle.**

APPLIC. *It is required to determine the diameter* CF, *of the circle* ACBF, *circumscribed to the triangle* ABC, (*fig.* 81.)

Determination. If from C, one of the angular points of the triangle the diameter CGF, be drawn through the centre G, of the circle; from the same angular point C, of the triangle the perpendicular CD, being drawn upon the opposite side AB; and the AF, be joined, there result two rectangular triangles CBD, and CFA, having the angles at B, and F, equal, because they stand in the same segment, and the angle at D, a right angle, by construction; as also, the angle CAF, because it is in a semicircle; which, therefore, being similar, give the proportion

$$DC : CB = AC : CF$$

therefore, $CF = \dfrac{CB \times AC}{CD}$ = diameter of the circumscribed circle.

If the three sides of the triangle ABC, are supposed given, and the value of the perpendicular CD, is substituted in the above expression as it has been found by the preceding Problem, this result will undergo the following mutations:

$$CF = \frac{CB \times AC}{\frac{2}{AB}\left(P\,(P-AC)\,(P-BC)\,(P-AB)\right)^{\frac{1}{2}}} = \frac{CB \times AC \times AB}{2\left(P\,(P-AC)\,(P-BC)\,(P-AB)\right)^{\frac{1}{2}}}$$

This determination evidently admits also of the introduction of the superficial content of the triangle, instead of the perpendicular, for if the perpendicular is expressed by the surface of the triangle as follows by:

$$S = \frac{CD \times AB}{2}; \text{ which gives}$$

$$CD = \frac{2S}{AB}; \text{ the introduction of this value of the}$$

perpendicular, mutates the formula thus:

$$CF = \frac{CB \times AC}{\frac{2S}{AB}}$$

$$= \frac{CB \times AC \times AB}{2S}$$

where, therefore, the diameter of the circumscribed circle is determined by the three sides of the triangle and its superficial content.

Corol. The last formula furnishes evidently a determination of the content of a triangle by its three sides and the diameter of the circumscribed circle; for its mutation furnishes

$$S = \frac{CB \times AC \times AB}{2CF}.$$

§ 77. PROBLEM. **To determine the radius of the circle inscribed in a given triangle.**

APPLIC. *It is required to determine the radius* CG = CF = CE, *of the circle inscribed in the triangle* ABD, (*fig.* 61,) *the three sides of which are given.*

Determination. If from the centre *C*, of the inscribed circle, lines be drawn to the angular points *A*, *B*, and *D*, of the triangle, this will be divided into three triangles, all having the same perpendicular, namely, the radius of the inscribed circle, and the three sides are the sides of the partial triangles upon which they fall. Their sum gives, therefore, the content of the whole triangle; thus: (calling *R*=radius of the inscribed circle)

$$S = \frac{AB \times R}{2} + \frac{AD \times R}{2} + \frac{BD \times R}{2}$$

because $\frac{R}{2}$ multiplies each of the terms on the right side, this becomes $S = \frac{R}{2}(AB + AD + BD;)$

which, by the substitution of $P = \frac{AB + AD + BD}{2}$; which has been used heretofore for the half perimeter of a triangle, becomes $S = P \times R$,

from which is obtained, for the value of the radius of the circle inscribed in the triangle,

$$R = \frac{S}{P};$$

Or the radius of the inscribed circle is equal to the surface of the triangle divided by the half perimeter.

This expression admits various mutation by the substitution of the value of S, and even of the perpendicular.

Substituting for $S = (P(P-AB)(P-AD)(P-BD))^{\frac{1}{2}}$

it results $R = \frac{\left(P(P-AB)(P-AD)(P-BD)\right)^{\frac{1}{2}}}{P}$

$$= \left(\frac{(P-AB)(P-AD)(P-BD)}{P}\right)^{\frac{1}{2}}$$

Corol. By this expression for the radius of the circle inscribed in a triangle, the surface of the triangle can evidently also be expressed; thus, from the above is given

$$S = R \times P = \frac{R}{2}(AB + AD + BD).$$

§ 78. PROBLEM. **To determine the side of an equilateral triangle, by the radius of its circumscribed circle.**

APPLIC. *The equilateral triangle* ABD, (*fig.* 62,) *having for its circumscribed circle,* AGDEBF: *it is required to determine its side* AB, *by the radius of the circumscribed circle; that is, by* CA = r.

Determination. It has been shewn before, (§ 56,) that the side of the regular hexagon is equal to the radius of the

circumscribed circle; therefore, in the figure there is $AC = CB = CF = AF = BF$; and CF, and AB, cutting one another in K, perpendicularly: they bisect each other, as perpendiculars of isosceles triangles; by the § 36, is therefore obtained, according to r = radius of the circle:

$$AK^2 = r^2 - \left(\frac{r}{2}\right)^2 = r^2 - \frac{r^2}{4}$$

and r, being common factor; $AK^2 = r^2\,(1 - \frac{1}{4})$

$$= r^2\,\tfrac{3}{4}$$

which gives $$AK = \frac{r}{2}(3)^{\frac{1}{2}}$$

and therefore $$AB = 2AK = r(3)^{\frac{1}{2}}$$

the side of the equilateral triangle is therefore equal to the radius of its circumscribed circle, multiplied by the square root of three.

Corol. 1. If the radius of a circle is considered as unity, the side of the equilateral triangle inscribed in it, will be equal to the square root of three.

Corol. 2. The side of the equilateral triangle is incommensurable with the radius of the circumscribed circle, that is they have no common measure.

§ 79. PROBLEM. To determine the side of a square inscribed in a circle.

APPLIC. *The radius* = r, *of the circle* ABDE, (*fig.* 82,) *being given to find the side of the square* ABDE, *inscribed in the circle.*

Determination. The side of the square inscribed in a circle must be the chord of a right angle at the centre, because each of the sides must subtend the fourth part of four right angles; it is therefore, the hypothenuse of a right angled triangle, of which the two sides about the right angle are equal to the radius, which was here called = r; thence results by (§ 36,)

$$AB^2 = r^2 + r^2 = 2r^2$$

and $$AB = r\,(2)^{\frac{1}{2}}$$

Or the side of the square inscribed in a circle, is equal to the radius multiplied by the square root of two.

Corol. 1. This proves also, that the diagonal of a square is incommensurable with its sides; for, what here refers to the radius and the side of the square, holds equally when these radii are sides of a square, and the diagonal of it is represented by the side of this square.

Corol. 2. If the radius of the circle is considered as unity, the side of the inscribed square will be equal to the square root of two.

§ 80. PROBLEM. **The side of a regular poligon inscribed in a circle, being given, to find the side of the poligon of double the number of sides, inscribed in the same circle.**

APPLIC. *The regular poligon* ABIGFH, *being inscribed in a circle,* (*fig.* 83,) *of which* AB, *be one of the sides, it is required to determine* AE = BE, *the side of the poligon of double the number of sides.*

Determination. The AB, being a chord of the circle, the CE, bisecting this chord, bisects also the arc AEB, and all the angles at D, the point of intersection of the chord and the radius are right angles; the triangles $AED = BED$, because $AD = BD$, and DE, common, and the angles at D, right angles, thence $AE = BE$, and (by § 36)

$$AE^2 = AD^2 + DE^2$$

By the construction we have

$$DE = CE - CD$$

thence

$$AE^2 = AD^2 + (CE - CD)^2$$
$$= AD^2 + CE^2 + CD^2 - 2CE \times CD$$

By the right angled triangle CAD, is obtained

$$CD^2 = CA^2 - AD^2$$

which being substituted, gives

$$AE^2 = AD^2 + CE^2 + AC^2 - AD^2 - 2CE(CA^2 - AD_2)^{\frac{1}{2}}$$

and because $CE = AC$, this becomes

$$AE^2 = 2AC^2 - 2AC(CA^2 - AD^2)^{\frac{1}{2}}$$

$$= 2AC\left(AC - (AC^2 - AD^2)^{\frac{1}{2}}\right)$$

therefore $AE = \left[2AC\left(AC - (AC^2 - AD^2)^{\frac{1}{2}}\right)\right]^{\frac{1}{2}}$

And because $AD = \frac{AB}{2}$, the use of the whole chord gives

$$AE = \left[2AC^2\left(AC - \left(AC^2 - \frac{AB^2}{4}\right)^{\frac{1}{2}}\right)\right]^{\frac{1}{2}}$$

giving the expression of the value of one side of the poligon of double the number of sides by the radius of the circle and the side of the given poligon. The expression of the double radicals shews these sides to be always incommensurable to one another.

The radius $= AC$, being made unity, or $= 1$, and the value of AP, expressed in that unity, being called $= a$. This formula takes a general form, thus:

$$AE = \left[2\left(1 - \left(1 - \frac{a^2}{4}\right)^{\frac{1}{2}}\right)\right]^{\frac{1}{2}}$$

$$= \left[2 - 2\left(1 - \frac{a^2}{4}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}$$

$$= \left[2 - \left(4 - a^2\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}$$

Scholium 1. By substituting in the expression just found, the value of the side of the regular poligon of *six sides*, as found above, (§ 56,) to be equal to the radius of the circle, the sides of the poligon of 12 equal sides will be obtained, thus:

$$AE = \left[2AC\left(AC - \left(AC^2 - \frac{AC^2}{4}\right)^{\frac{1}{2}}\right)\right]^{\frac{1}{2}}$$

by $AC = r$, the preceding suppositions:

$$AE = \left[2r^2 - 2r\left(\frac{3r^2}{4}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}$$

$$= \left(2r^2 - 2r^2\frac{(3)^{\frac{1}{2}}}{2}\right)^{\frac{1}{2}}$$

$$AE' = r\left(2 - (3)^{\frac{1}{2}}\right)^{\frac{1}{2}} = \text{side of the regular dodecagon.}$$

Scholium 2. Substituting for AB, the value of the side of the square inscribed in the circle of the radius $= r$, the general expression becomes

$$AE'' = \left[2r\left(r - \left(r^2 - \frac{r^2\ 2}{4}\right)^{\frac{1}{2}}\right)\right]^{\frac{1}{2}}$$

$$= \left[2r^2 - 2r\left(\frac{r^2\ 2}{4}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}$$

$$= \left[2r^2 - r^2\,(2)^{\frac{1}{2}}\right]^{\frac{1}{2}}$$

$$AE'' = r\left[2 - (2)^{\frac{1}{2}}\right]^{\frac{1}{2}}$$

which, therefore, is the side of the regular octagon inscribed in the circle, the radius of which is $= r$.

Corol. The continuance by the substitution of this value of the side of a poligon in the formula from which it is obtained, will again continue the same law for the determination of the value of the poligon next double in number of sides.

§ 81. PROBLEM. **The side of a regular poligon inscribed in a circle being given, together with the radius of the circle: to find the side of the regular poligon of equal number of sides, circumscribed to the same circle.**

APPLIC. *The side of the regular poligon inscribed in the circle being* AB, (*fig.* 84,) *and* C, *the centre of the circle, the radius of which are,* CA = CB = CE; *it is required to determine by these data the side* GF, *of the poligon of an equal number of sides, the sides of which will touch the circle at* E.

Determination. The radius CE, being perpendicular to both the chord AB, and the tangent GF, these two lines are parallel, and therefore, the triangles ACB, and FCG, are

similar, as are also the two triangles into which each of these is divided by the perpendicular CE; thence is obtained the proportion

$$CD : CA = AB : FG,$$

whence,
$$FG = \frac{CA \times AB}{CD}$$

The right angled triangle BCD, furnishes an expression for CD, by the radius, and the side of the inscribed poligon, thus:

$$CD^2 = CA^2 - AD^2$$
$$= CA^2 - \tfrac{1}{4}AB^2$$

and
$$CD = (CA^2 - \tfrac{1}{4}AB^2)^{\frac{1}{2}}$$

this can be transformed by making CA^2, a common factor, and extracting the root of it, thus:

$$CD = CA\left(1 - \left(\frac{AB}{2CA}\right)^2\right)^{\frac{1}{2}}$$

and this inserted in the value just found for FG, gives:

$$FG = \frac{AB}{\left(1 - \left(\frac{AB}{2CA}\right)^2\right)^{\frac{1}{2}}}$$

This is, therefore, the value of the poligon circumscribed to the circle in which the poligon with the side AB, is inscribed, and the radical expression shews again that these two sides are incommensurable to one another.

§ 82. PROBLEM. To determine by approximation, the circumference of the circle, in parts of its radius.

Determination. By § 80. The side of a regular poligon inscribed in the circle, being given, the side of the regular poligon of double the number of sides has been determined; the law of the expression being general, as has been shewn; this successive duplication of the number of sides of a regular poligon may be carried to any desired approximation to the circumference of the circle itself, for which it is then accepted. So that having in the general formula of § 80, accepted the radius as unity, and the side of the poligon, with double the number of sides expressed in that unity, being called $= a$,

the successive values found for the poligon of double the sides being substituted for a, in the general equation, the next multiple is obtained. To express, therefore, the circumference of a circle in numbers, of which radius is unity, the choice of that poligon is to be made, which leads by the simplest numbers, and the quickest series of duplication, thence of approximation to the desired aim, and this is the hexagon. The general formula being then:

$$AE = (2 - (4 - a^2)^{\frac{1}{2}})^{\frac{1}{2}}$$

for $a =$ the value of the dodecagon being by the *Scholium* 1, (§ 80,) and under the supposition of the radius, which is there $= r$, accepted equal to unity; the resulting side of the poligon next double, or of 24 sides, will become:

$$24 \text{ sides} = (2 - (4 - (2 - (3)^{\frac{1}{2}}))^{\frac{1}{2}})^{\frac{1}{2}}$$

$$= (2 - (2 + (3)^{\frac{1}{2}})^{\frac{1}{2}})^{\frac{1}{2}}$$

Substituting this value for a, in the general formula, the result for the poligon of 48 sides, becomes:

$$48 \text{ sides} = [2 - (4 - (2 - (2 + (3)^{\frac{1}{2}})^{\frac{1}{2}}))^{\frac{1}{2}}]^{\frac{1}{2}}$$

$$= [2 - (2 + (2 + (3)^{\frac{1}{2}})^{\frac{1}{2}})^{\frac{1}{2}}]^{\frac{1}{2}}$$

This substituted in the general formula, gives, by exactly the same mutation for the poligon of

$$96 \text{ sides} = [2 - (2 + (2 + (2 + (3)^{\frac{1}{2}})^{\frac{1}{2}})^{\frac{1}{2}})^{\frac{1}{2}}]^{\frac{1}{2}}$$

This shews the simple law under which this approximation progresses, by the subtraction of the new value of a, from 4, the sign of the preceding subtraction is changed, and the subtraction of the whole radical from 2, takes again place, so that always the successive radicals being obtained, and subtracted from 2, the root is again to be taken, this gives, for instance, for the poligon of 1536 sides, the

$$AE = [2 - (2 + (2 + (2 + (2 + (2 + (2 + (3)^{\frac{1}{2}})^{\frac{1}{2}})^{\frac{1}{2}})^{\frac{1}{2}})^{\frac{1}{2}})^{\frac{1}{2}})^{\frac{1}{2}}]^{\frac{1}{2}}$$

The side of the poligon circumscribed to the circle, and corresponding in number, having been determined by § 81, it may be used jointly with the value of the inscribed poligon to take an arithmetical mean between them, which, though not equal to the arc of the circle corresponding to the poligon, approximates it however more still.

The approximation being carried as far as desired by this subdivision, it is evident that the product of the value thus ascertained, by the multiplication with the number indicating the sides of the poligon, will give the value of the circumference of the circle for the radius unity; as for instance, the last quoted result would be multiplied by 1536.

This approximation is obtained by methods still more expeditious derived from higher branches of Mathematics, to an accuracy surpassing all that may be needed in any practice, namely, to 148 places of decimals. It is usual to give it as for the *Diameter*, equal to unity, and to designate it always by the Greek π. The following is this number:

π = 3, 14159 26535 89793 23846 26433 83279 50288 41971
69399 37510 58209 74944 59230 78164 06286 20899
86280 34825 34211 70679 82148 08651 32823 06647
09384 46095 50582 26136.

This number representing the circumference for the *Diameter* equal to unity: the circumference of any circle will be found by multiplying it by double the radius, or calling this circumference = C, there will be:

$$C = 2r\pi$$

(*Note.*) The extent to which the approximation of π, is carried by decimals constantly changing, sufficiently shows the impossibility and inutility of, what is called, the *quadrature of the circle.*

§ 83. PROBLEM. **To determine the content or surface of a circle.**

Determination. The preceding Problems have shewn in the determination of the sides of the regular poligons always triangles resulting, having their equal angles opposite the sides of the poligon in the centre of the circle, therefore the content

of each triangle being the half product of its base and perpendicular altitude, the whole poligon will be equal to the sum of all these sides multiplied by the altitude, and divided by two. Thence the principles of calculating triangles and the determination of this sum of all the sides of the regular poligon approaching the circle as near as desired, which furnished the above number π, will give the content of the circle; again to the same degree of accuracy as the approximation of π has been brought.

Taking therefore again C = circumference and r = radius of the circle, the superficial content = S, will be:

$$S = \frac{rC}{2}$$

and if for C, the circumference the value above found,

$$C = 2r\pi,$$

is substituted the equation becomes:

$$S = \frac{2r^2}{2}\pi = r^2\pi.$$

Scholium 1. This shews by the squaring of the radius and the constant number π, that circles (like all the rectilineal figures heretofore considered when similar) are to one another in the ratio of the square of their radii or diameters, as for any other circle of the radius = R, there would be obtained:

$$S' = R^2\pi$$

and thence the proportion,

$$S : S' = r^2\pi : R^2\pi = r^2 : R^2.$$

Scholium 2. Any section of a circle is evidently calculable upon the same principles, and is such a proportional part of the whole circle as the arc is of the circumference of the circle, thus therefore the arc being expressed by a fractional part of π, which call = n, will be for the sector whose radius = r, by calling the arc = a

$$a = \frac{2r\pi}{n}$$

and because for the same reasons as above we have the segments surface: $s = \frac{ra}{2}$; by substituting for a, its value, this becomes $s = \frac{r^2\pi}{n}$ = content of the *sector*.

§ 84. PROBLEM. To find the content of a segment of a circle.

APPLIC. *Let the segment of a circle contained between the chord* AB, *and the arc* AEB, (*fig.* 84,) *be determined the radius of the circle being* $= r$, *the chord* $= b$, *the arc* $= a$, *and the altitude of the segment* DE $= d$.

Determination. The content of the segment $AEBC$, has been determined by the preceding Problem, and the content of the triangle ABC, is to be considered as known, by the general principles for calculating contents of triangles, the difference between these two, is the content of the segment $AEBD$. Under the denominations accepted there will be obtained:

$$\text{Content of the sector} = \frac{ra}{2} = \frac{r^2\pi}{n}$$

$$\text{Content of the triangle } ABC = \frac{b(r-d)}{2}$$

Thence the segment between chord and arc, or $ADBE$, will be their difference, or

$$C = \frac{r^2\pi}{n} - \frac{b(r-d)}{2}$$

As the quantities r, b, d, and n, are dependent on each other, they will be determinable by one another, taking the properties of the circle to aid the determination, two of them will determine the two others, this expression will, therefore, admit several variations of form, according to the data selected. Thus:

1. The right angled triangle CBD, gives by § 66, *Schol.* 4,

$$r + CD : DA = DA : DE$$

and substituting the denominations adopted, observing that: $CD = r - d$, there results the proportion

$$2r - d : \frac{b}{2} = \frac{b}{2} : d$$

whence $$\frac{b^2}{4} = (2r - d)\,d$$

and $$\frac{b}{2} = (2r - d)^{\frac{1}{2}}\, d^{\frac{1}{2}}$$

giving $$C = \frac{r^2\pi}{n} - (r - d)\,(2r - d)^{\frac{1}{2}}\, d^{\frac{1}{2}}$$

expression in which the determination of the altitude of the segment, besides the radius, determines the triangle to be subtracted.

2. From the same proportion as the preceding, the value of d, being determined by b, and r, these two denominations may be alone retained in the expression, for that purpose is obtained:

$$2\,r\,d - d^2 = \frac{b^2}{4}$$

and

$$d^2 - 2\,d\,r = -\frac{b^2}{4}$$

$$d^2 - 2\,d\,r + r^2 = r^2 - \frac{b^2}{4}$$

$$r - d = \pm\left(r^2 - \frac{b^2}{4}\right)^{\frac{1}{2}}$$

thence again by substitution:

$$C = \frac{r\,\pi}{n} - \frac{b}{2}\left(r^2 - \frac{b^2}{4}\right)^{\frac{1}{2}}$$

$$= \frac{r\,\pi}{n} - \left(\frac{4\,r^2}{b^2} - 1\right)^{\frac{1}{2}}\frac{b^2}{4}$$

3. From the same proportion the radius may be expressed by the chord and the altitude of the segment, between chord and arc; and the value so found may be introduced; thus the proportion gives by product of extremes and means,

$$\frac{b^2}{4} = 2dr - d^2 = (2r - d)\,d$$

$$2dr = \frac{b^2}{4} + d^2$$

$$r = \frac{b^2 + 4d^2}{8d}$$

which being substituted in the first general expression for the segment, gives

$$C = \frac{\pi}{n}\left(\frac{b^2 + 4d^2}{8d}\right)^2 - \frac{b}{2}\left(\frac{b^2 + 4d^2}{8d} - d\right)$$

$$= \frac{\pi}{n}\left(\frac{b^2 + 4d^2}{8d}\right)^2 - b\left(\frac{b^2 - 4d^2}{2.8.d}\right)$$

giving, therefore, an expression for the case when the chord and the altitude of the segment are given.

Note. In all these results it has not been possible to avoid the value of n, or the proportional part of the circumference $= \pi$ corresponding to the chord b, and the altitude d, of the segment, though they evidently depend on each other, but though in certain cases this value is easily expressed, it is not generally so; and as these calculations are more habitually made with means, which have not yet been treated of in this volume, the above may suffice upon this subject.

PART II.

GEOMETRY OF SOLIDS.

CHAPTER I.

DEFINITIONS.

1. A *solid* is a magnitude of extension in three dimensions, commonly designated by length, breadth, and thickness.

2. *Solids* are *bounded* by surfaces, either plane or curved.

3. A *line is perpendicular to a plane:* when it is perpendicular to every line that can be drawn, in that plane, through its point of intersection with the same.

4. A *plane is perpendicular* to another plane: when a line drawn in it perpendicular to the common intersection of the two planes, is perpendicular to that other plane.

5. The *angle* of a line with a plane is measured by the angle which it makes with a line in that plane; that passes through its point of intersection with that plane, and through that of all the perpendiculars that may be drawn from the line upon that plane.

6. The *inclination of planes* to one another: is measured by the *acute* angle made by two lines drawn in the two planes, from the same point of their common intersection, perpendicular to this common intersection.

7. A *line is parallel* to a plane: when its unlimited protraction can never meet that plane.

8. *Planes are parallel* to one another: when their unlimited extensions, in any direction whatever, can never meet.

9. A *solid angle* is formed: when more than two planes intersect one another in the same point, (as fig. 102.)

10. When through the sides of a plane figure, planes are laid: that cut one another in one point without that plane, the solid produced is called a *Pyramid;* the plane figure through the sides of which the planes are laid is called the *base;* and the point where they meet: the *vertex*, of the pyramid, (fig. 104.)

11. When through the sides of a plane figure parallelograms are laid that meet one another, and are again bounded by a plane figure, equal and parallel to the first plane: the solid resulting is called: *a Prism*, (fig. 105.)

12. When through the four sides of a square, squares, equal to the same, are placed, perpendicular to this plane, that touch one another, and are again terminated by another equal square: the solid resulting is called: a *Cube*, (fig. 120.)

13. If a *right angled plane-triangle* is made to revolve around one of the sides including the right angle: the solid generated by this revolution is called: *a Cone*, (*ABE*, fig. 112.)

If the generating triangle has the two sides about the right angle equal, each of them being half a right angle, the angle at the vertex of the cone will be a right angle, and the cone is said to be *right angled.*

The side of the triangle around which it revolves is the *Axis* of the *cone.*

The other side, including the right angle which in that revolution describes a circle at right angle to the axis, forms the circular *Base* of the cone.

The hypothenuse of the generating triangle describes the *inclined Surface* of the *cone.*

14. If a *right angled parallelogram* is made to revolve around one of its sides: the solid generated is called: a *Cylinder*, (*ABCD*, fig. 112.)

The side of the parallelogram that remains fixed is called the *Axis* of the cylinder

The side parallel to the axis will describe in its revolution the *curved surface of the cylinder.*

The two sides perpendicular to the axis will describe parallel circles, that form the *Bases of the cylinder.*

15. If a *semicircle* is made to revolve around its diameter, the solid generated is called a *Sphere.*

The diameter upon which the semicircle revolves is called: the *Axis* of the sphere, and its two intersections with the surface of the sphere are called: the *Poles.*

The semicircle will, in its revolution, describe the *Surface* of the sphere.

All the radii of the semicircle being equal, all *the radii of the sphere* generated by it *are equal,* and the centre of the semicircle will be the *centre of the sphere,* and equally distant from every point in the surface of the sphere.

16. All other solids take their denominations either from the planes by which they are bounded, or the angles that their planes form with each other.

17. Any solid bounded by planes, in number more than *six,* is generally denominated a *Polyedron,* (fig. 111.)

18. Solids are called *equal* when their *solid contents* are equal.

AXIOMS.

1. Two plane surfaces *cannot include a solid angle.*
2. Three plane surfaces *cannot include a solid content.*
3. Solids which *fill the same space are equal.*

CHAPTER II.

Of the relation of planes, and the straight lines referred to them.

§ 1. THEOREM. A straight line cannot lie partly in a plane, and partly without that plane.

APPLIC. *The straight line* ABC, (*figure* 85,) *cannot lie partly in the plane* EF, *and partly without it.*

Demonstr. Suppose the straight line lie partly within and partly without the plane *EF;* so that *AB*, be within, and *BD*, without that plane; in the plane of *EF*, protract the *AB*, to *C;* two straight lines that have one part common, are in the same straight lines, *ABC*, and *ABD*, having the part *AB*, common, are in the same straight line; but this is against the supposition of *BD*, being different from *BC;* therefore, a straight line cannot lie partly in a plane and partly without it; which was to be demonstrated.

Corol. The protraction of a straight line lies in the same plane in which that line lies.

§ 2. THEOREM. Two straight lines that cut one another, lie in one same plane; and three straight lines, that cut one another, lie in one same plane.

APPLIC. *The lines* BD, *and* CE, (*fig.* 86,) *cut one another in* A, *they will lie in the same plane; and if* BC, *be joined, the triangle* ABC, *lies in the same plane.*

Demonstr. A plane being laid through the straight line *BAD*, and by revolving it upon that line, placed so that it meets the point *C*, then the points *A*, and *C*, will both lie in that plane, and therefore also the straight line *CAE*, which passes through these points; that is, the lines *BD*, and *CE*, cutting one another in *A*, lie fully in the same plane. As therefore also the points *B*, and *C*, lie in that plane, the straight line *BC*, which joins them, lies wholly in that plane; there-

fore, the three lines *AB*, *BC*, and *CA*, lie in one plane, that is the triangle *ABC*, lies in one plane, as was to be demonstrated.

Corol. This is generally expressed by : three points determine a plane.

§ 3. THEOREM. **If two planes cut one another, their intersection is a straight line.**

APPLIC. *The two planes* CD, *and* EF, *cutting one another in the points* A, *and* B, *the line* AB, *is a straight line,* (*fig.* 87.)

Demonstr. The points *A*, and *B*, of the line of intersection, being in the plane *CD*, the straight line *AB*, that joins them, lies fully in that plane; the point *A*, and *B*, lying also in the plane *FG;* the straight line *AB*, that joins them, lies also fully in the plane *FG;* therefore, the straight line *AB*, lies fully in both these planes, or forms a line of common intersection, and it is the same straight line; otherwise, two straight lines could be drawn between the same points, which would include a space, which is impossible.

§ 4. THEOREM. **If a straight line is perpendicular to two straight lines, that cut one another in a plane, at their common intersection, it is perpendicular to any line drawn in that plane, through this point of common intersection; therefore to the plane passing through these lines.**

APPLIC. *If the line* AB, *is perpendicular, the two lines* DC, *and* FE, *at their point of intersection* A, (*fig.* 88,) *it will also be perpendicular to any other line, as* GH, *that intersects both these lines and the perpendicular in the same point* A; *and to the plane passing through the lines* CD, *and* EF.

Demonstration. From the point of intersection *A*, make *AC* = *AD;* and also *AF* = *AE;* join *CE*, and *DF*, and through *A*, in the plane in which the *CD*, and *EF*, lie,

draw a line, as HAG, intersecting the DF, and EC, in G, and H. Conceiving a plane laid through CD, and AB, draw in it the DB, and CB; and in another plane, laid through FE, and AB, join FB, and EB. In the plane determined by the CBE, in which therefore the H, lies, because it lies in the CE, join also the BH; and in the plane through DBF, in which the G, must also lie, draw the BG. By the construction has been made $AC = AD$, and the $AE = AF$; the angles CAE, and DAF, are equal as vertical angles; the third sides of the triangles FAD, and CAE, are therefore also equal, or $DF = CE$; and the angle $ACE = ADF$. In the two triangles CAH, and DAG, the angles at A, are equal, as vertical angles, the angle $ACH = ADG$, by the above; and the straight lines between them $AC = AD$; the triangles are therefore equal, and $CH = DG$; $AH = AG$. In the two triangles ADB, and ACB, the sides AC, and AD, are equal; the perpendicular AB, is common to both triangles, and the angles BAC, and BAD, are equal, being right angles; therefore, also, the third sides are equal, or $BC = BD$. In similar manner, the two triangles BAE, and BAF, are proved equal, by the qualities $AF = AE$; $BAE = BAF$, and AB, common; therefore, the third sides are equal, or $BF = BE$. Hence, in the two triangles BEC, and BDF, the three sides are equal each to each; namely, $BF = BE$; $BD = BC$; and $DF = CE$; therefore, also, the angles opposite to equal sides are equal, as $BCE = BDF$. These equalities united, give for the triangles BCH, and BDG, the equalities $CH = DG$; $CB = DB$, and $BCH = BDG$; that is, two sides and the included angle in the two triangles equal, therefore also the third side equal, or $BH = BG$. There results, therefore, that the two triangles BAH, and BAG, have their three sides respectively equal, as $AH = AG$; $BH = BG$; and BA, common; thence the angles opposite to equal sides are equal, that is, $BAH = BAG$; but these angles are adjacent angles in the plane through $GABH$, therefore they are right angles; and the BA, is perpendicular to the GH, drawn in the plane through DC, and

FE, at their common point of intersection. The same would be the case with any other line drawn through *A*, in the same plane; therefore, *AB*, is perpendicular to the plane passing through *DC*, and *FE*.

Corol. 1. Any plane laid through a line perpendicular to another plane, is perpendicular to that plane; because it passes through the perpendicular and the line of intersection of the two planes which form any one of the planes produced by the preceding construction, and the perpendicular *AB*, is perpendicular to the common intersection *A*. And inversely: if a line, that is drawn in one plane perpendicular to the common intersection which it makes with another plane, is also perpendicular to that other plane, the two planes are at right angle to one another.

Corol. 2. If two planes pass through the same line perpendicular to a third plane, this line forms their common intersection, and the two planes are both perpendicular to that third plane. Inversely: the common intersection of two planes perpendicular to a third plane, is perpendicular to that third plane, as the intersections of the planes *ADBC*, and *AFBE*, formed in the Theorem.

§ 5. THEOREM. If three straight lines that cut one another in one point, are each of them perpendicular to a fourth straight line, these three lines lie in one plane.

APPLIC. *Let the three straight lines* AB, AD, *and* AC, *that meet one another in* A, *be each at right angle to* AE; (*figure* 89,) *they are in the same plane.*

Demonstr. Suppose they were not all three in the same plane, but that one of them lie out of the plane determined by the two others, as for instance *AB;* then, because two lines that meet one another determine a plane, if a plane be laid through *AD*, and *AC*, the *AE*, being perpendicular to these lines, will be perpendicular to the plane passing through these lines; if a plane be laid through the lines *AE*, and *AB*, this plane will be perpendicular to the plane through *AD*, and *AC;* and the common intersection of the two planes will be perpendicular to *AE*, because any line in that plane through *A*, is perpendicular to *EA*. Suppose *AF*, to be this common inter-

section, then will the angle *EAF*, be a right angle; but, by the supposition also, the *EAB*, in the same plane *EAF*, is at right angle with the *EA;* that is, the angles *EAB*, and *EAF*, are equal; but, by the anti-thesis they are unequal, which is contradictory; therefore, the third line *AB*, does not lie without the plane in which the *AD*, and *AC*, lie; that is, the three lines that are perpendicular to the fourth line *AE*, at their point of meeting, lie in one and the same plane.

§ 6. THEOREM. **If two lines are perpendicular to the same plane: they are parallel to one another.**

APPLIC. *The two lines* AB, *and* DC, *being perpendicular to the same plane, passing through the points* B, *and* C, *they are parallel to one another,* (*fig.* 90.)

Demonstr. Draw the straight line *BC;* from a point *A*, in the perpendicular *AB*, draw the *AC*, to the point where the other perpendicular meets the plane; in the plane upon which these lines are perpendicular draw the *CE*, perpendicular to *BC*, and equal to *AB;* join *BE*, and *AE*. In the triangles *ABC*, and *BCE*, the *AB* = *CE*, the *BC*, common to the two triangles, and the angles *ABC*, and *BCE*, are right angles; therefore the third lines *AC*, and *BE*, are equal also; the two triangles *ABE*, and *ACE*, have the sides *AB* = *CE; BE* = *AC;* and the *AE;* common to both triangles; they are therefore equal, and the angles *ABE*, and *ACE*, are equal, being opposite to the line *AE*, which is common to the two triangles; but *AB*, being at right angle upon the plane through *BCE*, the angle *ABE*, is a right angle; therefore also, the angle *ACE*, is a right angle; and because the *DC*, is perpendicular to the plane through *BCE;* the three lines *BC*, *AC*, and *DC*, are at right angle with the line *EC*, at their point of intersection *C;* therefore they lie in the same plane; and because the points *A*, and *B*, lie in that plane, also the *AB*, lies in it, because the two lines *AB*, and *BC*, lie in the same plane, and make with the *BC*, two right angles, on account of their perpendicularity to the plane, these angles

being the two inner angles of two lines with the cutting line *CB*, on the same side of it, the lines *AB*, and *DC*, will be parallel to one another; as was to be demonstrated.

Corol. If three lines are perpendicular upon every two sides of a triangle, at the angular points of it, they will be perpendicular to the triangle, and parallel to one another; and if planes are laid through every two of them, these planes will be perpendicular to the triangle.

§ 7. THEOREM. **If two lines are parallel, and one of them is perpendicular to a plane, the other is also perpendicular to the same plane.**

APPLIC. *The lines* AB, *and* DC, (*fig.* 90,) *being parallel, and the* AB, *perpendicular to the plane through* CB; *the* CD, *will also be perpendicular to the same plane.*

Demonstr. The same construction as in the preceding Theorem, the supposition of *ABC* = *ABE* = a right angle: gives by the perpendicularity of *EC*, upon *BC*, and *AC*; therefore to the plane in which the parallels *AB*, and *DC*, lie; or *ECD* = a right angle; and because the *AB*, and *DC*, are parallel, and the *BC*, cuts them, it makes with them the angles *ABC*, and *BCD*, together equal to two right angles; the angle *ABC*, being a right angle, the *BCD*, is a right angle also; the line *DC*, makes therefore the angles *DCB*, and *DCE*, equal to right angles; these lines being drawn in the plane upon which *AB*, is perpendicular, and being met at their point of intersection *C*, by the *DC*, under right angles at *C*, this *DC*, is perpendicular to the plane passing through these lines; that is, upon the same plane upon which *AB*, is perpendicular; as was to be demonstrated.

Corol. If on one of the angular points of a triangle a line be perpendicular to the triangle, lines through the two other angular points, that are parallel to the same: are also perpendicular to the same triangle.

§ 8. PROBLEM. **From a given point without a given plane: to draw a perpendicular to that plane.**

APPLIC. *Let the point* A, *be given without the plane* IK, (*fig.* 91,) *it is required to draw from it a straight line perpendicular to that plane.*

Solution. From the given point *A*, draw a line *AB*, to the plane *IK;* through the point *B*, intersection of this line with the plane, draw in the same plane the *DBC*, perpendicular to *AB;* and through the same point *B*, draw in the same plane the *BH*, perpendicular to *DBC;* from the given point *A*, draw the *AE*, perpendicular upon the *BH*, this line *AE*, will be perpendicular to the plane *IK;* as was required.

Proof. Through the point *E*, draw the straight line *GEF*, parallel to *DBC;* the *CB*, being perpendicular to the *BA*, and *BE*, by construction, is perpendicular to the plane through the *AB*, and *EB*, in which also *AE*, lies; and because the *GEF*, is parallel to *DC*, it is also perpendicular to the plane through *ABE*, therefore the angle *AEF*, is a right angle; the angle *AEB*, is a right angle by construction; therefore, the straight line *AE*, is perpendicular to the two lines *BH*, and *FG*, at their point of intersection in the plane *IK;* therefore it is perpendicular to that plane; as was required to be done.

Scholium 1. From one point without a plane, only one line can be drawn perpendicular to that plane; otherwise there would be two perpendiculars possible upon the same straight line.

Scholium 2. The perpendicular being the shortest distance from a point to a straight line, it is also the shortest distance from a point to a plane; thence it is, (like the perpendicular from a point to a straight line,) used to indicate the distance of a point from a plane.

§ 9. PROBLEM. From a point given within a plane, to draw a line perpendicular to that plane.

APPLIC. *From the point* A, *in the plane* EF, (*fig.* 92,) *it is required to draw a line perpendicular to that plane.*

Solution. Take any point without the plane, as *B*, and from it draw a line *BC*, perpendicular to the plane *EF;* through the point *A*, draw the straight line *AD*, parallel to *BC;* the *AD*, will be the perpendicular required.

Proof. The *BC*, being perpendicular to the plane *EF*, and the *AD*, parallel to it, the two lines *BC*, and *AD*, will be equally perpendicular upon the plane *EF*, by the foregoing proposition; therefore, from the point *A*, in the plane *EF*, there has been drawn the straight line *AD*, perpendicular to that plane; as was required.

Scholium. From a point within a plane only one line can be drawn perpendicular to that plane; because, only one straight line can be drawn through one point parallel to a straight line.

§ 10. THEOREM. When two straight lines are parallel to a third straight line, they are parallel to one another; though they be not in the same plane.

APPLIC. *The two lines* AB, *and* CD, *being each parallel to the* EF, *though all three be not in the same plane, they are parallel to one another,* (*fig.* 93.)

Demonstr. From any point *G*, in the line *EF*, draw, in the plane in which *EF*, and *AB*, lie, the *GH*, perpendicular to *EF;* from the same point draw, in the plane in which the *EF*, and the *CD*, lie the *GI*, perpendicular to the same *EF*. The *EGF*, being perpendicular to the *GH*, and *GI*, at their point of intersection, the *EGF*, is perpendicular to the plane passing through *IGH*. The *AB*, and *CD*, being each parallel to *EF*, are perpendicular to the same plane *IGH;* and because they are perpendicular to the same plane, they are parallel to one another; as was to be demonstrated.

Corol. 1. Any number of straight lines that are parallel to one and the same straight line, are parallel to one another, though they lie in different planes; for between them all the same reasoning of proof takes place.

Corol. 2. If, through every two of such parallel lines, planes are laid, these lines will form the common intersections of these planes.

§ 11. THEOREM. If a straight line is perpendicular to two planes, these two planes are parallel to one another.

APPLIC. *Let the line* AB, *be perpendicular upon the two planes* DE, *and* GF, (*fig.* 94,) *these two planes will be parallel to one another.*

Demonstr. Suppose they were not parallel, and that extended they will intersect one another, suppose in the straight line *IK;* in this common intersection, take any point, as *C,* join *CA,* and *CB,* then will *ACB,* form a triangle; because the line *AB,* is perpendicular to the plane *DE,* it makes right angles with any line meeting it in that plane; thence *BAC,* is a right angle. Because *AB,* is at right angle upon the plane *FG,* the angle *ABC,* is a right angle; therefore, in the triangle *ABC,* the angles at *A,* and *B,* are both right angles, which is impossible; therefore, the planes *DE,* and *GF,* do not meet, or intersect one another, in the line *IK;* therefore these planes are parallel; as was to be demonstrated.

Corol. Any number of planes through which the same straight line passes perpendicularly, are parallel to one another.

§ 12. THEOREM. When two parallel planes are cut by a third plane, their common intersections with that plane, will be parallel lines.

APPLIC. *Let the parallel planes* AB, *and* CD, *be cut by the third plane* EFHI, (*fig.* 95,) *their common intersections* EI, *and* FH, *with that plane, are parallel lines.*

Demonstr. If the lines of intersection, *EI,* and *FH,* are not parallel, they must cut one another, when protracted, suppose in *G;* as they are both in the same cutting plane *EH;* then because the straight line *EI,* lies in the plane *AB,* its protraction *IG,* lies also in that plane, (by § 1,) and in like manner because *HF,* lies in the plane *DC,* the *HG,* lies also in that plane, therefore the two planes *AB,* and *DC,* have the point *G,* common, or they cut one another; which is against the supposition that they be parallel; therefore, the straight lines *EI,* and *FH,* common intersections of the third plane with the two parallel planes, are parallel to one another; as was to be demonstrated.

§ 13. THEOREM. **When two straight lines, that cut one another, are parallel to two other straight lines, also cutting one another, though they are not in the same plane with the two first: the two planes passing through each pair of the cutting lines, are parallel to one another; and the angles which the cutting lines make with one another, are equal.**

APPLIC. *The two straight lines* AB, *and* AC, *that cut one another in* A, *being parallel to the two straight lines* DF, *and* DE, *that cut one another, but are not in the same plane with them,* (*fig.* 96,) *the plane passing through* CAB, *will be parallel to the plane passing through* EDF; *and the angles* CAB, *and* EDF, *will be equal.*

Demonstr. From the point of intersection *A*, of the two straight lines *AB*, and *AC*, draw the *AG*, perpendicular to the plane in which the *DE*, and *DF*, lie; from *G*, draw in this plane the *GH*, and *GI*, parallel to the two straight lines *DE*, and *DF;* by supposition and construction the *AC*, and the *GH*, are both parallel to *DE*, and so the *AB*, and *GI*, to *DF*, therefore the *GH*, is parallel to *AC*, and the *GI*, parallel to *AB;* the *AG*, being perpendicular to the plane through *EDF*, the angles *AGH*, and *AGI*, are right angles; on account of the parallel lines *AC*, and *GH*, the two angles *HGA*, and *CAG*, are equal together to two right angles, therefore the angle *GAC*, is a right angle; the same: on account of the parallels *GI*, and *AB*, and the right angle *AGI*, the angle *GAB*, is a right angle; therefore the straight line *AG*, is at right angles, or perpendicular to the two planes passing through *EDF*, and *CAB*, therefore these two planes are parallel to one another.

The two parallels *DE*, and *GH*, meeting the two parallels *DF*, and *GI*, respectively, their intersection, if supposed protracted, will make equal angles, that is, the angle *EDF*, will be equal to the angle *HGI*. If planes are supposed to be laid through the two pair of parallel lines *GH*, and *AC*, and through

GI, and *AB*, the perpendicular *AG*, will be the common intersection of these planes; the angles of these parallel lines with this perpendicular, being right angles, they are perpendicular upon the common intersection of the two planes; therefore each pair of these lines equally measures the inclination of these two planes, therefore they are equal to one another, or $BAC = IGH$; and because *IGH*, is equal to *EDF*, the angles *EDF*, and *CAB*, are equal to one another; therefore, when two lines cutting one another in one plane, are parallel to two other lines cutting one another in another plane, the angles which these lines make in the one plane, is equal to the angle made by the lines meeting in the other plane; as was to be demonstrated.

§ 14. THEOREM. **If straight lines be cut by parallel planes, they will be cut proportionally.**

APPLIC. *Let* GH, LK, *and* MN, *be three parallel planes, (fig.* 97,) *the lines* AB, *and* CD, *which they cut, are cut proportionally by them.*

Demonstr. Join *AD*, *AC*, and *BD*, and draw also the lines *EI*, and *IF*, in the plane *KL*, between the points of intersection of that plane with the given lines; the planes *KL*, and *MN*, being parallel, their intersections with the plane through the two lines *AB*, and *AD*, are parallel, that is *EI*, is parallel to *DB*; the same reasoning applied to *AC*, and *IF*, gives these lines also parallel to one another; therefore the triangles *ABD*, and *AEI*, are similar, as also the triangles *DAC*, and *DIF*; therefore their sides are proportional, thence results:

$$AE : EB = AI : ID$$

and $AI : ID = CF : FD$; therefore,

also, $AE : EB = CF : FD$,

that is: the given lines *AB*, and *CD*, are cut by the three planes proportionally; as was to be demonstrated.

CHAPTER III.

Of Solid Angles and the equality of Solids.

§ 15. THEOREM. If a solid angle be contained by three plane angles, any two of them are together greater than the third.

APPLIC. *Let the solid angle at* A, (*fig.* 98,) *be contained by the three plane angles,* GAF, GAH, *and* FAH; *any two of these angles will be together greater than the third.*

Demonstr. If the three angles are equal, it is self-evident that two of them are together greater than the third. If they are unequal, let FAG, be the greatest, then shall it be less than the two angles GAH, and HAF, together; make the angle $FAE = HAF$, the AE, will fall between the AG, and AF, because the angle FAH, is less than GAF. In AH, take any point, as D, and make $AE = AD$; through E draw a line cutting the two lines AF, and AG, in C, and B, join CD, and BD. In the two triangles ACD, and AEC, the lines $AD = AC$, and the angle $EAC = DAC$, by construction, the side AC, is common to both the triangles; therefore, they are equal to one another, and $CE = DC$. In the triangle BCD, the two lines BD, and DC, are together greater than the third line BC, taking on both sides the $CE = CD$, leaves the BD, greater than the part BE, of the line BC. In the two triangles ABD, and ABE, there is $AD = AE$, the AB, common to both triangles, and the third side BD, greater than BE; therefore, the angle opposite to BD, is greater than the angle opposite to BE, that is, GAH, is greater than GAE; therefore, the sum of the angles $GAH + HAF$, is greater than the sum of the angles $GAE + EAF$, that is greater than GAF, as was to be demonstrated.

§ 16. THEOREM. The sum of all the plane angles that contain a solid angle, is always less than four right angles.

APPLIC. *The sum of the three plane angles* BAD, DAC, *and* BAC, *forming the solid angle at* A, (*fig.* 99,) *are together less than four right angles.*

Demonstr. The sum of the three angles of the three triangles, including the solid angle *A*, is together equal to six right angles; the three angles of the triangle *BCD*, are together equal to two right angles, by the preceding proposition, two of the plane angles about one solid angle, are together greater than the third, therefore there is *ABD* + *ABC*, greater than the *CBD*; *ACB* + *ACD*, greater than *BCD;* and *ADC* + *ADB*, greater than *BDC;* therefore the sum of the angles of these side planes together is greater than two right angles; if therefore this sum is subtracted from the six right angles formed by the sum of all the side triangles, the remainder will be less than four right angles; as was to be demonstrated.

If the solid angle is formed of any number of plane angles, the same reasoning holds good for any, because there will be always the difference between the sum of the two angles of the side triangles greater than the third angle, which would form the angle of the poligon in the sides of which they terminate, and these will be in the same number as the sides of the poligon, the angles of which are always twice as many right angles as the figure has sides wanting four right angles; therefore, by their subtraction there must remain less than four right angles.

(*Remark.*) Like in the measure of the inclination of planes only, the acute angle is made use of; so also in solid angles only the acute solid angles are used, (habitually.)

§ 17. PROBLEM. A solid angle, included by three plane angles being given: to find the inclination of the planes that include the solid angle.

APPLIC. *Let the solid angle at* A, (*fig.* 100,) *formed by the planes* CAB, CAD, *and* BAD; *be given: it is required to find the inclination of the planes* ACB, *and* ACD, *to one another.*

Solution. In the common intersection AC, of the two planes take any point, as E, and from it draw the EF, perpendicular upon the AC, in the plane ACB, intersecting the AB, in F, draw also the EG, in the plane CAD, perpendicular to the same AC, which will cut the AD, in G; join FG. With the three sides EF, EG, and FG, construct the triangle efg; the side fg, being equal to FG, the angle e, which is opposite to it will be the angle of inclination of the two planes CAB, and CAD.

Proof. The angle made by the perpendiculars EG, and EF, in the two planes, perpendicular upon their common intersection AC, measures the angle of inclination of the two planes that intersect one another in AC, therefore the angle GEF, is the angle of inclination of these planes; the triangle egf, being constructed with the lines EF, EG, and FG, is equal to the triangle EGF, therefore the angle at e, which is opposite to $gf = GF$, will be equal to the angle of inclination of these planes.

Scholium. If only the three plane angles forming the solid angle were given, the angle of inclination between any two will be found by the same operation as above described, upon two cutting planes at their line of intersection, and having laid off the hypothenuse so determined, as were here the AG, and AF, upon the sides of the third plane angle, join the line there corresponding to FG; the three lines forming the triangle efg, and thence the angle $e = GEF$, will be obtained.

§ 18. PROBLEM. **Two plane angles forming part of a solid angle, and the inclination of their planes being given: to find the third plane angle forming the solid angle with the two others.**

APPLIC. *Let the two plane angles* BAC, *and* EDF, *be given, the sides of their intersection being* AB, *and* DE; *and the angle of inclination of these two planes equal to* HGI, (*fig.* 101,) *it is required to determine the angle* M, *of the third plane, which will complete the solid angle with the two others.*

Solution. In AB, and DE, the lines representing the common intersection between the two given planes, take AN, and DP, equal to one another; at the points N, and P, make the NQ, and PR, perpendicular to these lines, they will cut the AC, and DF, in Q, and R; upon the sides of the given angle of inclination of the two planes, HGI, make $GS = NQ$; and $GT = PR$; join the TS. With the three straight lines AQ, DR, and ST, describe the triangle MLK, the angle at M, opposite to the side $LK = ST$, will be the angle of the third plane forming the solid angle with the two others, which it was required to determine.

Proof. The perpendiculars NQ, and PR, are the sides of the angle G, formed, or given, as the inclination of the planes BAC, and EDF, because this angle is measured by the inclination of these perpendiculars; the line ST, drawn between their extremities, is the line corresponding to these two points, between the lines of intersections of the two given planes with the third plane, and the two hypothenuses AQ, and DR, of the right angled triangles ANQ, and DPR, form the two sides of the third plane including the angle sought, to which the side ST, is opposite. Therefore the triangle MLK, constructed with these three lines will have the angle M, opposite to $LK = TS$, equal to the angle of the third plane forming the solid angle with the two given planes; as was to be proved.

Corol. This and the preceding Problem shew how a solid angle is determined, either by three plane angles given, or by two plane angles and the inclination of these planes to one another; by which the third plane angle is determined, provided the solid angle is contained by three planes; if it is contained by a greater number of planes every two will determine a diagonal plane, which combined with an adjacent plane will determine a second diagonal, until the sides are all taken into account.

§ 19. THEOREM. **If, in two solid angles, contained each by three plane angles, either two of the plane angles, together with the angle of the inclination of these**

planes, are equal, respectively; or the angles of the three adjoining planes are equal each to each, respectively, and equally situated to one another; the two solid angles are equal to one another.

APPLIC. *In the two solid angles* A, *and* E, (*fig.* 102,) *let the angle* BAC = FEG; *the* BAD = FEH, *and the angle of inclination of the planes* BAC, *and* BAD, *be equal to that of the planes* FEG, *and* FEH; *then the third plane* CAD, *will be equal to the third plane* GEH; *and the solid angles* A, *and* E, *will be equal.*

Demonstr. 1. Lay the plane *BAD*, upon the plane *FEH*, so that *AB*, fall upon *EF*, and *A*, upon *E;* then because *BAD* = *FEH*, the line *AD*, will fall upon *EH;* because the angle of inclination of the two planes *BAC*, and *BAD*, is equal to that of the two planes *FEG*, and *FEH*, the plane *BAC*, will also fall upon the plane *FEG;* and because *BAC* = *FEG*, the line *AC*, will also fall upon the line *EG;* the two lines *AC*, and *AD*, coincide with the two *EG*, and *EH*, they will also include equal angles, that is, the third planes of these solid angles will coincide, thence be equal; and the solid angles *A*, and *E*, will fully coincide with one another, therefore be equal; as was to be demonstrated.

2. If the three plane angles forming the solid angles *A*, and *E*, are equal, respectively, and equally situated, the inclination of their planes must be equal between each of the planes, for in § 17, has been shewn, how, by the three given planes of a solid angle, the inclination of any pair of planes to one another is determined; therefore, if the three plane angles of two solid angles are equal, their inclinations will be equal, and thence the solid angle at *A*, equal to the solid angle *E*.

Scholium. If the number of plane angles including a solid angle, is more than three, then the solid angles will be equal, or determined, when one less than the number of planes are equal in the two, together with the angles which these planes make with one another; that is, two less angles than the number of planes, including the solid angles, determine all the solid angles.

§ 20. PROBLEM. At a given point in a given straight line, to construct a solid angle equal to another solid angle.

APPLIC. *The solid angle* A, (*fig.* 103,) *being given, to construct at the point* B, *upon the straight line* BC, *a solid angle equal to it.*

Solution. In one of the lines of intersection of the planes including the solid angle *A*, take any point, as *D*, and from it draw the *DF*, perpendicular to *DB*, in the plane *DAL*, intersecting the line *AL*, in *F*; from the same point *D*, in the plane *DAK*, draw the *DE*, perpendicular upon *AD*, cutting the *AK*, in *E*. From the point *B*, in the straight line *BG*, make *BG* = *AD*; and at *G*, make the perpendicular *GH* = *DF*, draw *BHN*; through *GH*, lay a plane perpendicular to the plane *GBN*, and in it make the angle *HGI* = *FDE*; and the line *GI* = *DE*, through the points *B*, and *I*, draw the *BIM*; the planes laid through the lines *BG*, and *BN*; the *BG*, and *BM*; and the *BM*, and *BN*, will form at the point *B*, a solid angle equal to the given solid angle *A*; as required.

Proof. The line *BG*, being perpendicular to the plane through *GH*, and *GI*, the plane *GHI*, is perpendicular to *GBN*; and the angle *HGI*, is, by construction, equal to *FDE*, or the inclination of the two planes *DAL*, and *DAK*; the *GB*, being equal to *AD*, the *GH*, to *DF*, and the angles between these equal lines being right angles, as they are perpendicular to each other, the third side *BH*, is also equal to *AF*, and the plane angles which they make at *DAF*, and *GBH*, are equal; by the same reasoning, the angles *DAK*, and *GBM*, are also proved equal; therefore, also, by the last §, the third plane angle *MBN*, must be equal to the third plane angle *KAL*; and therefore the solid angles at *B*, has been constructed on the line *BG*, at the point *B*, equal to the solid angle *A*; as was to be proved.

§ 21. THEOREM. In a solid angle formed by three planes, the sum of the angles of inclination of the planes

is more than two right angles, and less than six right angles.

APPLIC. *In the solid angle formed by the three planes* CAB, CAD, DAB, (*fig.* 100,) *the sum of the angles of inclination of these planes is more than two, and less than six right angles.*

Demonstr. The angle of inclination of these planes is measured by the angle of the lines EF, and EG, drawn perpendicular to the common intersection AE, of the two planes; these lines meet the intersections AB, and AD, of the third plane under different angles; the triangle FEG, thus formed, has for the sum of its angles two right angles. If at the other angular points of this triangle the AB, and AD, were perpendicular they would be parallel, and not meet in A. (§ 6.) If the angles of inclination of the planes were less than two right angles, they would not form a triangle, and the three planes could never meet. Therefore, as this sum can be neither *equal* to, nor *less* than, two right angles, it must be *more* than two right angles. If the sum of the three inclinations were six right angles they would either be all three equal, or they would be unequal; if they were equal, each of them would be equal to two right angles; that is, the planes would be in continuance of each other, (by proposition 5,) as a perpendicular upon their intersection would be equally perpendicular to both planes, and thus all the planes would lay in one. If these inclinations of the planes were unequal, some must be more than two right angles, therefore present an acute angle towards the opposite side, and again not meet in one vertical point; which is against the supposition.

Corol. If three planes that form a solid angle be at right angle to one another, they form a solid right angle. If the solid angle is formed by more than three planes: then, as the same reasoning holds good for every solid angle formed by any three planes, the sum of the inclinations must equally be more than two right angles, and as the number of planes determines the number of in-

tersections, which, by the above, must all together be less than as many times two right angles as there are planes, the sum of the angles of inclinations of the planes must be *less than twice as many right angles as the solid angle has sides.*

§ 22. THEOREM. If two equal solid angles are each intersected by a plane, at equal distances from the solid angular point, in each of the corresponding lines of intersection; the two solids formed will be equal to one another in all respects.

APPLIC. *Let the two equal solid angles* A, *and* E, (*fig.* 104,) *be intersected by the equal triangular planes* BCD, *and* FGH, *at the respectively equal distances* AB = EF; AC = EG; *and* AD = EH, *the two solids resulting will be equal in all respects.*

Demonstr. The solid angles *A*, and *E*, being equal, will coincide when applied one to the other, (by § 19,) the distances of the intersection of the third planes *BCD*, and *FGH*, being respectively equal, as : *AB* = *EF*; *AC* = *EG*, and *AD* = *EH*; also, the points *B*, *C*, and *D*, will coincide with the points *F*, *G*, and *H*, and the triangles *BCD*, and *FGH*, will coincide; therefore, the two solids *ABCD*, and *EFGH*, will coincide in all parts; they will therefore be equal in all respects, as was to be demonstrated.

Corol. 1. If to the triangles *BCD*, and *FGH*, were added in the same plane, the triangles *BID*, and *FKH*, the same reasoning would apply to the four planes meeting in *A*, and *E*, and the two quadrilateral figures *BCDI* = *FGHK*.

Corol. 2. As by definition, (10,) solids formed as above, are called *Pyramids:* this Theorem is generally expressed by : pyramids bounded by equal planes equally situated, are equal.

Scholium. If two of the planes uniting in a solid angle, as *ABC*, and *ABD*, having their common intersection in *AB*, (figure 105,) be protracted, and in them the lines *CK*, and *DI*, are drawn through the angular points *C*, and *D*, of the triangular base, parallel to their intersection *AB*; and through the vertex *A*, a plane be laid parallel to *BCD*, cutting these lines in *K*, and *I*, then

will be: $AB = CK = DI$, and the sides of the triangle BCD, equal to the sides of AKI, respectively, the figures BK, KD, IB, will be parallelograms. The solid resulting, will be a triangular prism; and if the same be done with two equal pyramids, as $ABCD$, and $EFGH$, the resulting prisms will be equal to one another, as the pyramids have been, because the solid angles are equal, and the planes that include them are equal. If the base of the pyramid be any polygon instead of the triangle, the same result will evidently follow for the prisms generated in the same manner, as that is only a repetition of the same process and result.

§ 23. THEOREM. **If three pair of parallel planes intersect one another mutually: the solid produced, will be bounded by six parallelograms, of which every two opposite one's will be equal and similar; and the solid angles that are diagonally opposite to one another, are equal. The solid resulting, is called a *Parallelepiped.***

APPLIC. *Let the two planes* ADGE, *and* BCFH, *be parallel to one another, and intersect the two parallel planes* ABCD, *and* EHFG, *and also the parallel planes* ABHE, *and* DCFG, (*fig.* 106;) *these planes will also mutually intersect one another in parallel lines, forming parallelograms that bound a solid called a* Parallelepiped. *Every pair of opposite parallelograms will be equal, and equiangular; and the solid angles diagonally opposite to one another, as* A, *and* F, *will be equal.*

Demonstr. The two parallel planes $ABCD$, and $EHFG$, are cut by the plane $ADGE$, the intersections AD, and EG, are therefore parallel; the same planes being cut by the plane $BCHF$, the intersections BC, and HF, are parallel also; and because the planes $EADG$, and $HBCF$, are parallel the intersections with the plane $ABCD$; that is, AD, and BC, are parallel; thence the AD, BC, HF, and EG, are all parallel; and because parallels between parallels are equal, they are also equal to one another. In the same manner, the parallel planes $ABHE$, and $DCFG$, that are cut by the plane $ADEG$, make

the intersections AE, and DG, parallel, and because of the parallelism of the planes $ADGE$, and $BCFH$, the lines AE, DG, CF, and BH, are also parallel, and equal to one another. In the same manner, the result of the comparison of the parallel planes $ABHE$, and $DCFG$, cut by the parallel planes $ABHE$, and $EHFG$, that are also parallel to one another, make their common intersections AB, DC, GF, and EH, equal, and parallel. The planes inclosing this solid, will therefore be parallelograms, of which the opposite one's will always be equal, and similar to one another. Because the AD, and AE, are parallel to the BC, and BH, though not in the same plane, the angles which they form with one another, are equal, that is, $EAD = HBC$; and by the principle from Plane Geometry: that the opposite angles in parallelograms are equal, the angle HFC, is equal to HBC, hence to EAD; in like manner is proved, that the other angles are equal, which form the solid angles A, and F, thence they are formed by the same number of equal plane angles; but these angles are diagonally opposite to one another in the solid; the same reasoning will apply to any other pair of opposite solid angles; therefore, in the figure $ABCDGEHF$, the opposite planes are parallel and equal, as well as similar, and the opposite solid angles are equal; this solid is called a *Parallelepiped*, and is bounded by six parallelograms, of which every two that are opposite, or parallel, are equal in all respects.

Corol. 1. If one of the solid angles is rectangular, all the others will be rectangular, because when one angle in a parallelogram is a right angle, all the others are right angles; and the solid will be a rectangular parallelogram.

Corol. 2. If all the sides of the parallelograms that bound the parallelepiped are equal, the parallelograms becoming squares, the solid bounded by them will be a cube, by definition 12.

§ 24. THEOREM. **If a parallelepiped is cut by a plane passing through the diagonals of two opposite parallelograms, this plane will divide the parallelepiped into two equal and similar triangular prisms.**

APPLIC. *The parallelograms* ABCD, *and* EFGH, *that are opposite to one another in the parallelepiped* DF, (*fig.* 107,) *being divided by the diagonals* AC, *and* EG; *the plane* ACGE, *laid through them, will divide the parallelepiped into two equal and similar triangular prisms* ABCGEF, *and* ACDEGF.

Demonstr. By the preceding proposition, the opposite parallelograms in any parallelepiped are equal, and similar, (by § 29,) the diagonal of any parallelogram divides it into two equal triangles; these diagonals being in the same plane with the sides of the parallelograms of which they are the diagonals, and these parallelograms being parallel, the diagonals are parallel also; and the figure *ACGE*, is a parallelogram. Thence in the two prisms formed by the diagonal plane, every solid angle is bounded by three planes that have alternately the same angles, and form the same solid angles: as the angle at *D*, and *H*, in the one, equal to the angle at *F*, and *B*, in the other prism; the same is the case with respect to the partial angles at *C*, *G*, *A*, and *E*; therefore, the two prisms, in which the parallelepiped has been divided, are bounded by an equal number of equal planes, and equal solid angles, they are therefore equal and similar; as was to be demonstrated.

§ 25. THEOREM. **Parallelepipeds upon the same bases, and terminated by the same parallel planes, are equal to one another.**

APPLIC. *Let the two parallelepipeds* ABGFCHED, *and* QPNRCHED, *be upon the same base* CHED, (*fig.* 108,) *and between the same parallel planes, passing through their common base* CHED, *and a plane laid through the two opposite planes* AG, *and* QN, *the parallelepipeds will be equal to one another.*

Demonstr. In the plane in which the parallelograms *AG*, and *QN*, lie, protract the sides *FG*, and *AB*, of the one parallelogram; and also the sides *QR*, and *PN*, of the other; these lines will intersect one another, and form again a parallelogram

IKLM, which will be equal to each of the two others, because with each of them it lies between the same parallel lines *AL*, *FM*, and *IQ*, *MP*. Protract the parallel planes *DG*, and *CB*, till to *M*, and *L*; protract likewise the parallel planes *CR*, and *EP*, to *I*, and *M*, these planes having the base *CE*, common, and also the parallelogram *KM*, will intersect one another in the lines *CK*, *HL*, *DI*, and *EM*, because these points are common to both pair of parallel planes, these lines will all four be equal to one another, as parallels joining parallels of the equal parallelograms *CE*, and *KM*, and these will lie in the protraction of the side planes of the given parallelograms, as, *ID*, and *ME*, in the protraction of *FGDE*; *EHLM*, in the protraction of *EHPN*. The *FG*, *IM*, *AB*, and *KL*, being equal as opposite sides of the same or equal parallelograms, if to each of them be added, the *GI*, or the *BK*, the lines *FI*, *GM*, *AK*, and *BL*, will all be equal; and because of the parallel planes *ALC*, and *FME*, and the equality of the parallelograms that form the sides of the parallelepipeds between the same parallels, the prisms which these triangles form with them are equal; that is, that *AFDCKI* = *BGEHLM*; if, therefore, from the whole solid contained in *AFDCHLME*, we take away the first prism, *AFDCKI*, there remains the parallelepiped *HCKLMEDI*, and in taking away the other prism from the equal whole figure, there remains the parallelepiped *AE*, first given; this parallelepiped is therefore equal to the parallelepiped *CM*. If the same reasoning is made between the parallelepipeds *CM*, and *CN*, which remain after the prisms *QPCHKL* = *RDEIMN*, are subtracted from the whole of the solid content *DEIMPQCH*, the parallelepipeds *CN*, and *CM*, are equal to one another; each of the two parallelepipeds *CN*, and *CG*, being therefore equal to *CM*, they are equal to one another; that is, parallelepipeds upon the same base, and between the same parallel planes, are equal.

Corol. 1. The prisms which here resulted by this construction between the two parallel planes, have proved themselves to be equal in whatever direction the base of them is taken; it may

therefore immediately also be concluded, that prisms set upon equal bases and between the same parallel planes, are equal; which besides follows in consequence of this Theorem united to the preceding one.

Corol. 2. This proposition extends naturally to any parallelepipeds of equal bases, and between the same parallels, though the bases might not be the same, or similar; for, as by Plane Geometry, it is shewn how parallelograms may be changed into others under different angles and sides, but equal to the same. This might be led through the same demonstration as the Theorem, but being more of long words than difficult comprehension, the mere attentive inspection of the (figure 109,) will shew the successively equal parallelepipeds which come into consideration. What is said of parallelepiped, applies equally to prisms, they are always to be considered as halves of parallelepipeds.

Corol. 3. The distances of parallel planes being measured by the perpendicular between them, these results are generally expressed thus: The solidity of parallelepipeds is equal to the product of their base into the altitude.

§ 26. THEOREM. Pyramids between the same parallel planes, and upon the same or equal bases, are equal to one another.

APPLIC. *The pyramids* ABCD, *and* EFGH, *stand upon the equal and similar triangles* BCD, *and* FGH, (*fig.* 105,) *being in the same plane, and the vertices* A, *and* E, *are in the same plane parallel to that in which the triangles are, these pyramids shall be equal.*

Demonstr. Complete the two prisms upon the triangular bases in the manner shewn, (in § 22, *Schol.*) Then will prisms so formed be equal to one another, because they are the halves of parallelepipeds between the same parallel planes, whose base would be the double of the triangle, (by § 24 and 25.) The pyramids standing upon the same triangular base as the prisms, are a certain determined part of them, equal in both; therefore, these pyramids will also be equal to one another, as was to be demonstrated.

§ 27. THEOREM. Every pyramid is the third part of the prism, of equal base and altitude.

APPLIC. *The triangular pyramid,* ABCD, (*fig.* 110,) *shall be the third part of the prism* AEFBCD, *that is between the same parallel planes, or of the same altitude, and upon the same triangular base.*

Demonstr. Having completed the prism *AEFGCD*, corresponding to the pyramid upon the same triangular base, draw the diagonal *EC;* the sides of the pyramid *AD*, and *AC*, form diagonals to the two parallelograms *BE*, and *BF;* these diagonals divide, therefore, each of these parallelograms into two equal triangles. In the pyramid *ABCD*, the triangle *ABD*, being considered as base, and the point *C*, as vertex, the pyramid *AECD*, will have the base *AED* = *ABD*, and its vertex will be in the same point *C;* therefore, these two pyramids have equal bases and altitudes; they are, therefore, equal to another. By the diagonal *EC*, the two triangles *ECD*, and *EFC*, are equal, and taking them as bases of the triangular pyramids *ADCE*, and *AFCE*, having their vertices common in *A*, the two pyramids which they form are again equal; as the pyramid *ADCE*, has been proved equal to *ABDC*, the three pyramids are all equal, as *ABDC* = *ADCE* = *AFCE*. These three pyramids complete the whole triangular prism that stands upon the same triangular base with the pyramid *ABCD*, therefore, they are together equal to it; or the pyramid *ABCD*, is the third part of the prism *ABCDEF*. The same conclusion will hold good for pyramids of any poligonal base; as by the decomposition of the poligonal base in the number of triangles it contains, every pyramid admitting the above proof, the whole number of them admits also the same, and the principle applies equally to it. So that, generally, *a pyramid* is the third part of the prisms of equal base and altitude; as was to be demonstrated.

§ 28. THEOREM. A polyedron is equal to another polyedron, when all its solid angles, and the rectilineal

figures which form them, are equal, and similar to one another, and similarly placed.

APPLIC. *In the polyedrons* BACDEFGHI, *and* bacfghi, (*fig.* 111,) *the solid angles of the corresponding letters being respectively equal, and the planes including the equal angles, equal and similar, and similarly placed towards one another: the polyedrons shall be equal.*

Demonstr. The two solids having an equal number of equal solid angles, bounded by three equal and similar planes, each of these will form with the plane that joins the other ends of their lines of common intersection triangular pyramids that will be respectively equal to one another, because they could be made to coincide in all respects, as: the angles $A = a$; joining the diagonals $FC = fc$, which will make the triangles AFC, and afc, equal, and joining also $FI = fi$, will form the two equal pyramids $AFCI = afci$, bounded by equal and similar planes, equally situated; in a similar manner will be formed the equal pyramids $ABIF = abif$, and so all the others; therefore, also, their sum total, or the poligons themselves, will be equal to one another; as was to be demonstrated.

§ 29. THEOREM. **Cones and cylinders, of equal base and altitude, are equal.**

APPLIC. *Let* AB, *and* FG, *be two equal circles,* (*fig.* 112,) *and let upon them be constructed the cylinders* ABCD, *and* FGHI, *and the cones* ABE, *and* FGI, *having the same altitude, or that may be between the same parallel planes, then shall the cylinders be equal to one another, and also the cones.*

Demonstr. In Plane Geometry it has been shewn how the surface of a circle can be determined like that of any other poligon; and that circles of equal radii are equal; hence the circles AB, and FG, presenting by the supposition equal bases to the cylinders $ABCD$, and $FGHI$, their altitude being the same, their solid content will always be represented by the product of the base into the altitude, and therefore be equal

under the products of equal bases and altitudes, like any other prisms with equal bases and altitudes. In like manner, the equal circles present equal bases to the cones *ABE*, and *FGI*, which stand upon them, and between the same parallels, they are therefore equal parts of the cylinders upon the same bases; therefore, also, the cones are equal to one another; as was to be demonstrated.

Scholium. As the cone can be presented by the triangular bases that might be formed by two radii, and their included arc, and the triangular pyramids formed upon them coincide all in one vertical point, the solidity of each such pyramid being (by § 27,) the third part of the prism of equal base and altitude; it follows also that cones are the third part of the cylinders of equal base and altitude.

§ 30. THEOREM. **Spheres of equal radii are equal, and their surfaces are equal.**

Demonstr. In Plane Geometry has been proved: that circles of equal radii are equal. By the definition of the sphere as generated by the full revolution of a semicircle around its diameter, it is also evident that each point of surface being equally distant from the centre of the generating circle, when the radii of two circles thus revolving are equal: the spheres which they generate must be equal; and the surfaces that the equal circumferences describe, and that are the surfaces of these spheres, will be equal also; as was to be demonstrated.

CHAPTER IV.

Of the similarity and proportionality of Solids.

§ 31. THEOREM. If a triangular pyramid be cut by a plane parallel to its base: the common intersections of the side planes are cut proportionally; the pyramid cut off by the intersecting plane, and the triangles that include it, will be equiangular to the corresponding parts of the whole pyramid, and the two bases will be to one another as the squares of their distances from the vertex.

APPLIC. *Let the triangular pyramid* ABDC, (*fig.* 113,) *be cut by the plane* GEF, *parallel to* DBC; *then will the sides* AB, AC, *and* AD, *be cut proportionally; the angles at* E, *will be equal to the angles at* B, *and so* F, *and* G, *to* C, *and* D; *the triangular bases* BCD, *and* EFG, *will be to one another as the squares of the parts of the intersections of the sides from the vertex, or as the squares of the perpendiculars from the vertex.*

Demonstr. The two planes BCD, and EFG, being parallel and cut by the third plane ABC, in BC, and EF, these lines of intersection are parallel to one another; the triangles ABC, and AEF, will be similar, and the sides are cut proportionally, thus:

$$AC : AB = AF : AE,$$

and the other two planes for the same reason furnish the equal proportions,

$$AC : AD = AF : AG, \quad \text{and}$$
$$AB : AD = AE : AG;$$

therefore, these triangles are equiangular to one another, and the third sides are also proportional; that is,

$$EF : BC = FG : CD = GE : DB;$$

thence the triangles BCD, and EFG, are similar, and their

corresponding angles equal; therefore, all the angles including the solid angle at B, are equal to all those including the solid angle at E; and the same will be with the other corresponding angles, F, and G; to C, and D, the two pyramids $ABCD$, and $AEFG$, are therefore equiangular, and all their sides are proportional. Similar figures being to one another as the squares of the homologous sides, the two triangles forming the bases of the cones, give the proportion,

$$BCD : EFG = BC^2 : EF^2$$

But the above $BC : EF = AB : AE$, gives also,

$$BCD : EFG = AB^2 : AE^2 = AC^2 : AF^2 = AD^2 : AG^2$$

that is: these bases are in the same ratio to one another, as the squares of the distances of their angular points from the vertex of the cone. If a perpendicular be drawn from the vertex A, upon the two bases, as APp, the two parts of this perpendicular, Ap, and AP, will be in the same proportion as these lines of intersection cut by the same parallel planes, (§ 14,) therefore these planes are also to one another in the ratio of the squares of the perpendicular distances from the vertex: or,

$$BCD : EFG = AP^2 : Ap^2 .$$

By the similarity of the triangles forming together the solid angles at B, and E, they are included by equal plane angles similarly situated, therefore they are equal to one another; in the same manner, also, the solid angles at F, and C, are equal, and those at G, and D; therefore the two pyramids $AEFG$, and $ABCD$, are equiangular, and therefore similar; for similar solids are such as have equal solid angles.

Corol. This proposition applies equally to pyramids of any poligonal base, and therefore, also, to cones, by the considerations given, (§ 29.) In a triangular pyramid, every angular point may be taken as a vertical angle, and a plane laid parallel with the opposite plane, will cut it in two similar pyramids.

§ 32. THEOREM. **If a parallelepiped is cut by a plane parallel to one of its sides: it will be divided into two parts, which will have the same ratio to one another, and to the whole, as the respective bases have.**

APPLIC. *Let the parallelepiped* AEGCDHFB, (*fig.* 114,) *be cut by the plane* IKML, *parallel to the sides* EC, *or* DF, *then will the parallelogram be cut in two,* AM, *and* BM, *that will be to one another as their bases* EM, *and* MD; *and to the whole as their respective bases to the base of the whole* AH.

Demonstr. If the sides *GM*, and *MH*, of the parallelogramic base *EM*, and *DM*, are commensurable, they can be divided each in a certain number of equal parts; it has been proved in Plane Geometry: that the parallelograms *EM*, and *DM*, which are upon these two bases, and between the same parallels, are to one another as their bases. If planes be laid through these divisions, as *ag*, *me*, and also through the points *b*, *d*, *c*, &c. in the parallelepipeds *GI*, and *MB*, parallel to the side planes *AG*, and *BH*, these planes will divide the two parallelepipeds in so many parallelepipeds upon equal bases, thence equal to *AGag*, as are equal divisions in the *GM* and *MH;* the ratio of the two parallelepipeds will therefore be the same as that of the number of these equal parallelepipeds, that is : equal to the number of equal bases into which the bases of the parallelepipeds have been divided.

If the parallelogramic bases, or the sides forming the bases of these parallelograms, are not commensurable, the parallelepipeds are also in the ratio of their bases. For, suppose the parallelepiped *GI*, was to the whole *GB*, not as *GL*, to *GF;* but as *GL* to *Gy*, or as *GM*, to *Gx;* the *Gy*, and *Gx*, being smaller than *GF*, and *GH;* taking a subdivision of *Gm*, smaller than *xH*, and laying it off on *GH*, till one of the points falls in *z*, between *x*, and *H;* then the parallelepiped *GI*, and *Az*, give the proportions:

$$Az : GI = Ez : GL, \text{ but the supposition is:}$$
$$BG : GI = Ex : GL,$$

Which gives, therefore, the ratio of the larger parallelepiped *BG*, to another parallelepiped *GI*, smaller than the smaller parallelepiped *Az*, which is impossible; therefore, whether the bases are commensurable or not, there will be obtained,

$$\left\{\begin{matrix}\text{Parallelepiped}\\ GI,\end{matrix}\right\} : \left\{\begin{matrix}\text{Parallelepiped}\\ GB,\end{matrix}\right\} = \left\{\begin{matrix}\text{Base}\\ GL,\end{matrix}\right\} : \left\{\begin{matrix}\text{Base}\\ GF,\end{matrix}\right\}$$

and by subtraction,

$$GI : GB - GI = GL : GF - GL, \text{ or}$$
$$GI : MB = GL : MF.$$

That is, if the parallelepiped CB, is cut by the plane MI, parallel to the side planes GA, and BH; the resulting parallelepipeds are to one another as their bases.

Scholium 1. Every prism cut by a plane parallel to one of the triangular or poligonal bases, will also be cut in parts that are to one another in the ratio of the intersections of the parallelogramic sides: as the portions of the partitions of the parallelograms form the bases of the partitions of the parallelepipeds; and a prism is the half of the parallelepiped that would be formed by the completion of the parallelogramic bases, from the triangle forming the base of the prism, and the intersected parallelograms of the sides.

Scholium 2. Similar solids having all a determined constant ratio to the parallelepiped of equal base and altitude with them; this Theorem generalises in a similar manner, as those relating to figures of equal altitude or base in Plane Geometry; that is: *similar solids, of equal altitude, are to one another as their bases*, which, by the considerations made, (§ 29,) applies equally to cones and cylinders; and is equally true inversely, so: *similar solids, of equal bases, are to one another as their altitudes.*

§ 33. THEOREM. Similar parallelepipeds are to one another in the ratio of the cubes of their homologous sides.

APPLIC. *The parallelepipeds* AI, *and* cm, *being similar, that is, the solid angles at* B, *and* b, *being equal, and the sides* AB, *and* bc, *homologous sides, then shall the parallelepipeds be to one another in the ratio of the cubes of the homologous sides* AB, *and* bc.

Demonstr. In the parallelepiped AI, protract the three sides, including the angle B, then will the vertical solid angle resulting be equal to the angle b, of the parallelepiped cm. In the protracted AB, take $BC = bc$, and also in the protracted

DB, the $BE = be$, and in the protracted GB, the $BF = bf$, complete the parallelpiped CM, by planes parallel to the planes passing through these three lines and the points C, F, and E, alternately; extend the side planes of these parallelepipeds, so as to form by their mutual intersection, the complementary parallelepipeds BH, and BK.

The parallelepiped CM, having its solid angle at B, equal to the solid angle b, of the parallelepiped cm, and the sides about these angles being equal in both the parallelepipeds, they are equal and similar. The parallelepipeds AI, and BH, are between the same parallel planes, they are, therefore, in the ratio of their bases, or the parallelograms AG, and GC; for the same reason, the parallelepipeds BH, and BK, are as their bases BO, and BL, and the parallelepipeds BK, and BN, as their bases BP, and BS. But these bases being parallelograms between the same parallels, respectively, they are, by Plane Geometry, as their bases, or as BA, to BC, and so on. We obtain, therefore, the following proportions:

Parallelepipeds. Parallelograms. Lines.

$$AI : BH = AG : GC = AB : BC$$
$$BH : BK = BO : BL = BG : BF$$
$$BK : BN = BP : BS = DB : BE.$$

Because the sides of the parallelepipeds are proportional, we have also:

$$AB : BC = BG : BF = BD : EB,$$

Thence by the compound ratio of the three first, we have:

$$AI : BN = AB : EB;$$

these sides being in continued proportion, the parallelepipeds are in the triplicate ratio of either one of them, as: $AB : BC$, that is, in the ratio of their cubes, which gives the ultimate proportion:

$$AI : BN = AB^3 : BC^3$$

and as the parallelepiped BN, is equal to the parallelepiped bn, also:

$$AI : bn = AB^3 : bc^3$$

that is: similar parallelepipeds are to one another as the cubes of their homologous sides; which was to be demonstrated.

Corol. 1. Triangular prisms being the half of the parallelepipeds of equal base and altitude, their ratio is also that of the parallelepipeds, that is, similar prisms are to one another as the cubes of their homologous sides; therefore, also, as the cubes of their altitudes, these forming likewise homologous sides.

Corol. 2. Pyramids being the third of the prisms of equal base and altitude, the conclusion goes also upon them: that is, similar pyramids are to one another as the cubes of their homologous sides, and as above also, as the cubes of their altitudes.

Corol. 3. In like manner, the proposition extends to all polyedrons, as they may be decomposed in as many triangular pyramids as they have side planes.

Corol. 4. The cones, cylinders, and spheres, having been shewn to have the same principles of equality as the polyedrons: they have of course also the same principles of proportionality; and we have thereby the general result: "*Similar solids are to one another in the ratio of the cubes of their homologous sides.*"

§ 34. THEOREM. If a sphere be cut by a plane, the intersection of it with the surface of the sphere, will be a circle, the radius of which will be a mean proportional between the two parts of the diameter at the point of intersection with that plane.

APPLIC. *Let the sphere of which the circle* AEBF, *is a great circle,* (*fig.* 116,) *be cut by the plane* LM, *the intersection* AHBG, *with the surface of the sphere, will be a circle, the radius of which* AD, *will be a mean proportional between* FD, *and* DE; *the centre of the sphere being* C.

Demonstr. From the centre *C*, draw *FCD*, perpendicular to the plane *LM*. From any point of intersection of the plane with the surface of the sphere, as *A*, draw *ADB*, through the point *D*, where the perpendicular meets the cutting plane, protracted till it cuts again the intersection of the plane with the surface of the sphere in *B;* join *CA*, and *CB;* because *A*, and *B*, are points in the surface of the sphere, the *CA*, and *CB*, are equal; in the two triangles *CAD*, and *CBD*, the angles at *D*, being right angles, the *CA* = *CB*, and *CD*, common, we

have $CB^2 - CD^2 = CA^2 - CD^2 = DA^2 = DB^2$; the point H, being taken also in one of the points of intersection of the plane LM, with the surface of the sphere, and CD, being perpendicular to the cutting plane, the angle CDH, is also a right angle, and $DH^2 = CH^2 - CD^2$, but CH, is equal to CB, and CA, as radii of the same sphere; therefore, $DA = DB = DH$, and so would any point taken in the intersection $AHBG$, of the plane with the surface of the sphere; therefore, this intersection is a circle. Any one of these radii, as DA, DB, &c. are at right angles upon the diameter FE, because this diameter is perpendicular to the circle $AHBG$, in its centre D; therefore we have by Plane Geometry, (§ 66,) the $DA^2 = FD \times DE$; that is, the diameter of the intersected circle a mean proportional between the two parts of the diameter of the sphere; which was to be demonstrated.

Corol. 1. If the cutting plane passes through the centre of the sphere, its intersection with the surface is a great circle of the sphere.

Corol. 2. The circle described on the surface of the sphere, by any plane intersecting it, is called a parallel circle; a perpendicular from the centre of the sphere upon its plane, passes through its centre, and inversely, the perpendicular upon its plane in the centre of the circle, will pass through the centre of the sphere. The two points, F, and E, where the perpendicular meets the surface of the sphere, are called the poles of the parallel, and they will be in the middle of the segments of the sphere formed by the intersecting plane.

§ 85. THEOREM. **The plane laid perpendicular at the end of the radius of a sphere, is a tangent to the sphere at that point, or it has only that point common with the sphere.**

APPLIC. *Let the plane* DB, (*fig.* 117,) *be laid perpendicular to the radius* CA, *at the point* A, *in the surface of the sphere, the* AFE, *being one of the great circles through* A, *the plane* DAB, *will be tangent to that circle; therefore, to the sphere.*

Demonstr. Because the *CA*, is perpendicular to the tangent plane *BD*, any other line drawn from *C*, to that plane like *CG*, will be greater than this perpendicular; therefore, the point *G*, that lies outside of the great circle *AEF*, and of any other that can be drawn upon the surface of the sphere; for all their radii are equal to *CA*, as the radius of the sphere is equal to that of any great circle of the sphere.

§ 86. PROBLEM. **From a given point in the surface of a sphere to lay a plane cutting the same, the parallel of which shall subtend a given angle at the centre of the sphere.**

APPLIC. *Let* A, *be a point in the surface of a sphere, the centre of which is* C, (*fig.* 118,) *it is required to lay a plane from it through the sphere, so that the parallel which it describes shall subtend at the centre* C, *a given angle* = M.

Solution. Through *A*, lay a plane, tangent to the sphere, an intersection of which be *BAH*, through the radius *CA*, and the *BAH*, lay another plane perpendicular to the plane passing through *BH*, which will form by its intersection with the surface of the sphere the great circle *ADEF;* in this plane draw the *AD*, cutting the circumference of this circle in *D*, and making with *AH*, the intersection of the two planes, an angle equal to *half* the given angle *M;* through *AD*, lay a third plane perpendicular to the great circle *ADEF;* this plane will describe upon the surface of the sphere the parallel which will contain at the centre *C*, an angle equal to the given angle *M*.

Proof. The plane of the great circle *ADEF*, being laid through the radius *CA*, perpendicular to the tangent plane, and intersecting it in the line *AH*, the angle *HAD*, will cut off from that great circle an angle equal to half the angle at the centre, that is *AD*, will be the chord of this circle, subtending at the centre an angle *ACD*, equal to the angle *M;* because it
uble the angle between the tangent and the chord, by
Geometry. A plane laid through this chord perpen-

dicular to the great circle will describe a parallel circle subtending the same angle at the centre, the perpendicular from the centre C, upon that plane will fall in the centre of the parallel circle, making all its radii equal (by § 34,) and subtending, at the centre C, angles equal to the angle DAH, therefore the parallel described by the cutting circle, will subtend at the centre C, an angle equal to the given angle M, as was required to be done.

§ 37. PROBLEM. To place a sphere so that its surface shall touch a given point in the surface of another sphere, and pass through a given point outside of the same sphere; or, to describe a sphere tangent to a given sphere.

APPLIC. *Let the point* B, *be in the surface of a sphere, the centre of which is* C, (*fig.* 119,) *and the point* A, *without it: it is required to lay a sphere so: that its surface shall pass through the points* A, *and* B, *and not cut the sphere in the surface of which* B, *lies; that is, be tangent to the sphere at the point* B.

Solution. Join AB, and draw through B, and the centre C, of the given sphere the indefinite straight line BCI; these two lines will determine a plane, which will intersect the sphere in the great circle BGF; bisect the AB, in D, and from D, in the same plane in which BA, and BI, lie, draw a perpendicular to AB, it will intersect the BI, in E; from E, as centre, with the radius $EA = EB$, draw the circle ABH; the sphere described by the revolution of this circle, upon the centre E, will be a sphere tangent to the given sphere.

Proof. The three points A, B, and C, determine a plane passing through the centre of the sphere and by its intersection forming in it the great circle BFG, which thence lies in that plane; the sphere which shall touch this given sphere must therefore have its centre in the line from the centre to the point of contact B, like in circles that touch one another in Plane Geometry. Considering the BA, in the same plane, as a

chord of a great circle of the sphere, intended to be made tangent to it, the centre of it must lie in the perpendicular upon the middle *D*, of this chord, in the same plane, and therefore in the intersection of these two lines, that is in *E;* and the distances *EA*, and *EB*, will be equal; therefore, from the centre *E*, drawing the circle *ABH*, with the radius *EA* = *EB*, it will be the great circle of a sphere which, by the revolution around one of its diameters, as around the line *BI*, or the centre *E*, will have no point common with the given sphere but the point *B*, that is it will be tangent to the given sphere; for if a plane be passed through *B*, perpendicular to the *BI*, it will lay fully outside of both spheres, because it is perpendicular to the end of the radius of both spheres equally.

Corol. 1. The two figures relating to this proposition, shew: that when the point of contact of the given sphere lies within the two tangents from the given point to the sphere, the tangent sphere lies without the given sphere, and when it lies in either of these points, or on the opposite side of them, in relation to the given point *A*, the sphere to be described, incloses the given sphere.

Corol. 2. If the given sphere is made to revolve around the radius *AE*, the point *B*, of the sphere described as above, will describe upon this sphere a small circle, which will be a parallel of the sphere.

CHAPTER V.

Application of Arithmetic to Solid Geometry.

§ 38. PROBLEM. To determine the solid content of a cube; and also its surface.

Determination. It is known by the elements of Arithmetic, that a *ratio* can be directly stated only between things of the same nature; therefore also the content of solids can be only compared to any one solid, taken as *Unity* of the kind. By the demonstration of § 34 the sides of two squares being n, and m, the cubes upon these sides will be to one another as $n^3 : m^3$; either one of these being considered unity, or, what is the same thing, measuring the other, we derive from it the principle that the content of any cube whatever is the cube of the side of the same, or if $n = 1$, we have the cube upon $m = m^3$.

In (fig. 120,) there will be in m, as many squares of lines $= n$, in the base of the larger square m, as sides n, in the sides of m, and upon these will be formed by the same n, taken in elevation as many cubes as squares in the base, repeated m, times; thence the bases being as $n^2 : m^2$; the repetition of this ratio in the supposition, gives the ratio $n^3 : m^3$.

The surface of the cube being the sum of the six surfaces which include the cube, the sum of the content of these, and in general, of all surfaces of solids, are to be calculated by the principles of Plane Geometry; these give in this individual case any of the squares representing the sides $= m^2$; therefore, the whole surface of the cube $= 6\ m^2$.

§ 39. PROBLEM. To determine the solid content of a parallelepiped, the base and altitude of which are given.

Determination. By the propositions upon the equality of parallelepipeds upon equal bases, and between the same pa-

rallel planes, we have seen that parallelepipeds, under any angle of the sides upon the plane of the base, are equal when their bases and altitudes are equal; therefore, any parallelepiped can be treated as having a base equal to the given base, and the planes upon these bases at right angle, limited by the parallel planes, determined by their altitude; therefore, the content of the parallelepiped, under whatever angle, will be equal to the product of its base into its altitude, or the perpendicular between the base and its opposite plane. So, that if b^2 represent the parallelogramic base, and h = altitude, the content of the parallelepiped will be $= b^2.h = C$, being a quantity of three dimensions in the same manner as in the preceding problem for the cube, the m^3; or the parallelepiped, of which all the sides are equal.

Scholium 1. The triangular prism having been proved to be half the parallelogram of equal base and altitude, when one of the parallelograms is taken for base, the solid content of a triangular prism will be the half of the product of the base into the altitude of the triangle, forming its opposite sides; that is, we would in that case have $= \frac{b^2\ h}{2}$ for b, and h, as bases, and (height or) altitude of the solid.

Scholium 2. As it has been proved by the general equality of similar solids, of equal base and altitude, that prisms of any poligonal base are equal when their bases are equal, and they are between the same parallel planes, that is, of the same altitude, if any prism is on a triangular or poligonal base whatever, and its plane opposite to the base is equal and parallel to it, the solid content of any such prism, will be like that of any parallelogram, equal to the product of its base in its perpendicular altitude, or we shall in this case also have, under the preceding acceptations, $b^2\ h$, for the solid content of the prisms.

Corol. 1. All solids being in a certain determined *Ratio* to the parallelepiped, it is evident: that this Ratio acts in every case like a constant multiplier, either whole or fractional; and that therefore all what relates to the content of solids, is dependent entirely on this constant factor, and the principles of their generation, given in the definitions.

Corol. 2. In all these cases it is evident, that the surfaces of the solids will be equal to the sum of the superficial content of all the parallelograms, or triangles including the solid.

§ 40. PROBLEM. To determine the solid content of a cylinder, and of a cone.

Determination. In Plane Geometry we have determined the superficial content of a circle, and obtained a quantity of surface, in all respects in the same manner as of the parallelogram, or any rectilineal figure whatever; therefore, we obtain also for the calculations of Solid Geometry, a superficial base of a prism of a circular figure, which, multiplied by its altitude, will give the solid content of the prism of a circular base, and of the altitude given by its perpendicular altitude.

I. If, therefore, we call r = radius of the circular base of a cylinder, and h = altitude of the same, we shall have, (by § 82, of Plane Geometry,) for the circumference of the circular base of the cylinder, $= 2r\pi$. Where π is the number determined in § 82, the superficial content of the circle will be $C = r^2\,\pi$, as before determined; and, therefore, any cylinder of the altitude $= h$, will have for its solid content:

$$S = h.r^2.\pi.$$

II. It has been proved, that the cone was the third part of the cylinder upon the same circular base and of the same altitude, therefore we have for the solid content of the cone the third part of the above content of the cylinder, under the same radius $= r$, and altitude $= h$, or for the cone,

$$P = \frac{h.r^2.\pi.}{3}$$

§ 41. PROBLEM. To find the content of the curved surface of a rectangular cylinder; and of a cone the perpendicular from the vertex of which falls in the centre of the circular base.

Determination 1. The circumference of the circle of the radius $= r$, forming the base of the cylinder was $= 2r\pi$; if there-

fore a plane is conceived to turn around this circle, terminated by a circle parallel to the base, and perpendicular upon the same, a parallelogram is formed, the base of which is the circumference of this circle and the altitude $= h$, that of the cylinder, this surface will therefore be determined like a parallelogram, and be the product of the circumference of the base into the altitude of the cylinder: or,

$$C = 2.h.r.\pi.$$

2. For the cone we have evidently the slent lines of the sides from the vertex to the circular base, all equal, therefore, radii of the same circle, as already evident from the principle of the generation of the cone, by which the hypothenuse is the side which describes this surface; therefore, the circumference of the base forms a part of the circumference of a circle, the radius of which is that slent side, and this circumference measures the arc, or the part of the circumference corresponding to it; this part of the content of the circle is, therefore, to be calculated upon the same principle as a whole circle would be, that is, by the product of the arc into half the corresponding radius, which is here half the slent side; if, therefore, this slent side is $= m$, the side surface of the cone becomes, by the above:

$$C' = m.r.\pi.$$

3. If m, was not given, and only the altitude $= h$, it would either have to be calculated from this and the radius of the circle, or its value substituted in the place of m, here. For this we have $m^2 = h^2 + r^2$; therefore, $m = (h^2 + r^2)^{\frac{1}{2}}$ which by substitution, would transform the above into:

$$C' = (h^2 + r^2)^{\frac{1}{2}} \; r.\pi.$$

§ 42. PROBLEM. To determine the solid content of a truncated pyramid, or cone, that is formed by the base and a plane cutting it parallel to that base.

APPLIC. *Let* A, *be the vertex of a triangular pyramid upon the base* BCD, *or* a, *of a cone upon the circular base,*

BC, (*fig.* 121,) *and let the pyramid, or the cone, be cut by the triangle* bcd, *or the circle* bc, *parallel to these bases; it is required to determine the solid content of the truncated pyramid, or cone, contained between these two planes.* (*This truncated part is called a Frustum.*)

Determination. By the preceding sections, it has been proved: that the solid content of a pyramid, or cone, is the third part of the product of its base into the altitude; the two pyramids *ABCD*, and *Abcd*, will therefore be equal, respectively, to the product of the base *BCD*, or *bcd*, into their respective perpendicular distance from the vertex *A*, or their perpendicular altitude; exactly the same will be the case with the two cones *aBC*, and *abc*. The difference between these two is evidently in both cases the solid content of the frustum, or the truncated pyramid, or cone; the planes forming the bases of the two pyramids, or cones, having the same vertex, being parallel to one another, they are to one another in the ratio of the square of their distances from the vertex, (by § 31.)

From these principles are deduced the following determinations, viz:

Calling; h = altitude of the frustum, or perpendicular distance between the two planes.

x = altitude of the whole pyramid, or cone; that is, perpendicular distance of the vertex from the base. Hence,

$x - h$ = altitude of the pyramid cut off.

B^2 = surface of the base of the frustum.

b^2 = surface of the upper plane terminating the frustum.

By these denominations, which apply equally to the pyramid and the cone, the solid content of the whole pyramid will be expressed according to the above principles, by:

$$S = \frac{B^2 . x}{3}$$

and the solid content of the part cut off above the upper plane, or parallel to the base, will be:

$$s = \frac{b^2.(x - h)}{3}$$

The difference between these two results, therefore, is the solid content of the frustum, which is therefore expressed by

$$F = S - s = \frac{B^2.x}{3} - \frac{b^2.(x - h)}{3}$$

In this expression, the x, or altitude of the pyramid, or cone, cut off above the frustum by the plane parallel to the base, is yet to be determined, from the principle which has been quoted, from § 31, that the bases of the pyramids, or cones, are to one another as the squares of their distances from the vertex, which gives the proportion:

$$B^2 : b^2 = x^2 : (x - h)^2$$
$$B : b = x : x - h$$
$$b.x = B.x - B.h$$
$$B.x - b.x = B.h = (B - b)\,x$$
$$x = \frac{B.h}{B - b}$$

This value of x, substituted in the above expression for the frustum, gives:

$$F = \frac{B^3.h}{3(B - b)} - \left(\frac{B.h}{B - b} - h\right)\frac{b^2}{3}$$
$$= \frac{B^3.h}{3(B - b)} - \left(\frac{B.h - h.B + b.h}{B - b}\right)\frac{b^2}{3}$$
$$= \frac{B^3.h}{3(B - b)} - \frac{b^3.h}{3(B - b)}$$
$$= \frac{h}{3}\left(\frac{B^3 - b^3}{B - b}\right)$$

and executing the division of the second part, actually:

$$F = \frac{h}{3}\left(B^2 + b.B + b^2\right)$$

That is: the solid content of the frustum of a pyramid, or cone, is equal to the product of the third of its altitude into

the sum of the two parallel planes, and a mean proportional between them. This result is evidently general for any figure presenting the base $= B^2$, and other parts similar. If therefore the B^2, is a surface of a circle, its expression by the radius $= R$, and the number $= \pi$, as by § 82, in Plane Geometry, is to be substituted for it in the formula, thus: designating the radii corresponding to the two circles BC, and bc, by R, and r, the conic frustum will be expressed by

$$F' = \frac{h}{3}\left(R^2.\pi + r.R.\pi + r^2.\pi\right)$$

$$= \frac{h.\pi}{3}\left(R^2 + r.R + r^2\right)$$

§ 43. PROBLEM. **To determine the superficial content of the sides of a truncated pyramid, or of a truncated upright cone.**

APPLIC. *In (fig.* 121,) *it is requested to determine the content of the quadrilateral figures* cbBC, bdDB, *and* cdDC, *which inclose the side of the pyramid; or the part of the curved surface of the truncated upright cone, contained between the two circles whose diameters are* BC, *and* bc, *that is the cone, of which the perpendicular from the vertex, falls in the centre of the circle, forming its base.*

Determination 1. The content of each side of the truncated pyramid is evidently the difference between the two triangles Abc, and ABC, and so on, forming the sides of the pyramid cut off, and of the whole pyramid; (the lines BC, and bc, are parallel, and the figure formed is called a Trapezium.) The content of each of the triangles will be equal to half the product of its base into the altitude, or perpendicular from the point A, upon that side of the triangle lying at the base of the pyramid; and because these triangles are similar, their altitudes are as their bases. These principles give the following calculation of their content:

Calling y = perpendicular from A, upon BC;

p = the perpendicular distance between the two lines

BC, and bc; the contents of the figure, or trapezoid, $BCcb$, will become:

$$C = \frac{BC \times y}{2} - \frac{bc \times (y - p)}{2}$$

From the similarity of the triangles ABC, and Abc, is obtained the proportion,

$$BC : bc = y : y - p,$$

whence y, will become determined by the two bases BC, and bc, and their perpendicular distance thus:

$$bc.y = BC.y - BC.p$$

whence, $$(BC - bc)\, y = BC.p$$

and $$y = \frac{BC.p}{BC - bc}$$

which substituted in the above expression for the value of y, gives:

$$C = \frac{BC^2.p}{2(BC - bc)} - \frac{bc}{2}\left(\frac{BC.p}{BC - bc} - p\right)$$

or, $$= \frac{BC^2.p}{2(BC - bc)} - \frac{bc\,(BC.p - BC.p + bc.p)}{2(BC - bc)}$$

by compensation,

$$C = \frac{p\,(BC^2 - bc^2)}{2(BC - bc)}$$

$$= \frac{p\,(BC + bc)}{2}$$

The same is evidently the result with any of the two parallel intersections of the sides of a pyramid by a plane parallel to the base, and the sum of all the trapezoids so formed will form the sides of the truncated pyramid of any base whatever.

(*Note.*) This gives at the same time the principle of calculation of any trapezoid whatever, the two parallel sides of which are BC, and bc.

2. In the cone the same reasoning applies to the circles forming the base of the two cones of which the truncated cone is the difference; for the parts of circles formed by the circumferences of the bases of the two cones will have the same angle,

thence be similar, and in the same ratio as their radii, that is, their slent distances from the vertex of the full cone; thence we have under the same denominations as above, and substituting for the circumferences of the circles forming the two bases, their values expressed by their radius, and the constant number π, calling R = radius of the greater, and r = radius of the smaller circle,

y = slent distance of the vertex from the base,

p = slent distance of the two circles; the circumferences of the circles forming the bases of the inclined surfaces, will be, (by § 82,)

$$2R\pi, \text{ and } 2r\pi,$$

thence these being multiplied by half the slent distance of the vertex, and their difference taken, the slent surface of the frustum of the cone will be:

$$C' = y.R.\pi - r.\pi.(y - p)$$

and because we have like in the preceding case,

$$R : r = y : y - p;$$

and all the substitutions and reductions follow, exactly as in any one of the single trapezoids of the pyramid, we have also the ultimate result of the same form, that is:

$$C' = \pi.p.(R + r)$$

for the surface of the conical part of the truncated cone the radii of the circles bounding it, being R, and r, and the perpendiculars passing through both centres; which is evidently only the case in what is called the upright cone.

§ 44. PROBLEM. **To determine the surface of a sphere.**

APPLIC. *The section of the sphere through its centre being determined by the great circle* ABbdgDeE, (*fig.* 122,) *the radius of which* be = r, *it is required to determine the surface of the sphere of which it is the section.*

Determination. Draw the diameter of the sphere *AG;* and contiguous to each other, perpendicular to this diameter, lay the two planes passing through *BE*, and *be*, parallel to each

other: so that their arcs Bb, and Ee, may be confounded with the tangent to the radius at the meeting of it with the surface of the sphere, in the same manner as this has been done in determining, in Plane Geometry, the circumference of the circle from its diameter, by the approximating poligon; then the radius CB, being drawn, CBb, will be a right angle, drawing the Bn, perpendicular upon pb, the PBn, will also be a right angle, and the triangle Bbn, will be similar, the triangle CBP, for the angles $bBn = CBP$, and $Bbn = BCP$; this furnishes the proportion:

$$PB : CB = Bn : Bb;$$

thence $$Bb = \frac{CB \times Bn}{PB}$$

Under the same supposition the part of the circumference bB, being allowed to be considered as a straight line by the principle of the calculation of the side surface of a truncated cone, ascertained in the last proposition, the side surface of this truncated cone, of which Bb, is the slent altitude, will be:

$$s = \pi Bb \times (PB + pb);$$

Upon the same ground of the near contiguity of the planes through EB, and eb, the PB, and pb, that is the two radii of the parallels can be considered as equal; by which this expression becomes:

$$s = 2\pi Bb \times PB,$$

placing in this expression the value of Bb, just found by the proportion, it becomes:

$$s = \frac{2\pi . CB \times Bn \times PB}{PB};$$

and by the compensation of PB, in numerator and denominator:

$$s = 2.\pi . CB \times Bn;$$

and because of the radius $CB = r$,

$$s = 2.\pi . r . Bn.$$

The $Bn = Pp$, is any minute part into which the radius may be divided: so that $\frac{r}{Bn} = \frac{r}{Pp}$, will represent this radius

expressed in the same minute units, or subdivisions, which have been used above.

This proposition is general, and applies equally to any part of the circumference of the sphere between two parallels, thence also to their multiples expressed by the same units; thence to the whole sphere, which would have $\frac{2r}{Bn}$, for its expression in units of Bn; therefore, the surface of the whole sphere will be equal to the product of these two quantities, that is:

$$S = 4\pi r^2 \frac{Bn}{Bn} = 4\pi r^2 .$$

As the surface of the circle of the radius r, has been found in Plane Geometry, (§ 83,) to be equal to πr^2, the comparison of these results shews, that the surface of the sphere is four times that of the great circle of the same sphere; and this result is evidently independent of all arbitrary supposition, being grounded entirely upon the determination of the circumference of a circle from its diameter; which, as has been shewn in its place, is obtained by an extreme approximation; the Pp, or distance of the parallel planes forming the subdivisions of the radius, or any part of the same, is exactly corresponding to the subdivision of the circle by the regular polygon, in which the circumference is subdivided by the approximation.

If it is desired to have this expression reduced to the diameter of the sphere, the value of $r = \frac{D}{2}$, is to be substituted for r, by which the expression will become:

$$S = \frac{4\pi D^2}{4} = \pi D^2 .$$

that is, the circumference of the great circle of the sphere multiplied by its diameter, is equal to the surface of the sphere.

Scholium 1. Every part of the surface of the sphere will equally be obtained, by the substitution of the value of the part of the radius to which it corresponds, in the units of the subdivision employed; which evidently will always disappear by the compensation that takes place in the multiplication: as for in-

stance, there will be, according to the above principles:

$$BAE = 2\pi.r.AP.\frac{Bn}{Bn} = 2\pi.r.PA.$$

in like manner the zone:

$$BEDd = 2\pi.r.Pf.$$

§ 45. PROBLEM. To determine the solid content of a sphere.

APPLIC. *Let the figure* (122,) *of the foregoing Problem, represent the sphere of which the solid content is to be determined.*

Determination 1. Having by the preceding Problem determined the surface of the sphere, it is evident that drawing radii from any three or more points of it, taken equally as minute subdivisions, as have always been done for this object, they will form the bases of pyramids, having their vertices in the centre of the sphere; and that therefore, the solid content of the sphere will be again equal to the sum of the solid contents of all these pyramids. Every pyramid being the third part of the product of its base into its altitude, the expression for the surface found above, being multiplied by the third part of the radius, will give the solid content of the sphere, that is:

$$C = 4\pi r^2.\frac{r}{3} \qquad \text{or,}$$

$$C = \frac{4\pi r^3}{3}.$$

2. If in the general expression, the diameter is desired to be introduced instead of the radius the value of $r = \frac{D}{2}$ or, $r^3 = \left(\frac{D}{2}\right)^3$ is to be substituted in it, which gives:

$$C = \frac{4\pi D^3}{3.8} = \frac{\pi D^3}{6}$$

Or the sixth part of the number π, given in Plane Geometry, would form a constant number, which multiplied into the cube of the diameter of a sphere, will give its solid content.

Scholium. Any sector of the sphere will of course be expressed by the part of the circumference to which *it* corresponds, so for instance for the sector from the centre *C*, resting upon the part

of the circumference cut off by EB, as above used, there will be obtained:

$$C = 2\pi r.AP.\frac{r}{3}$$

$$= \frac{2\pi r^2.AP}{3}$$

(*Remark.*) It may be observed by the preceding Problem, and by this: that in all cases where a surface is obtained, there will always be two lineal dimensions resulting, to be multiplied: as here, the r^2, or D^2; when a solid is determined, three lineal dimensions will be multiplied, as here, r^3, or D^3. Thus always denoting the nature of the result, in accordance with the calculation, and the nature of the subject, under consideration, or obtained in result.

§ 46. PROBLEM. **To compare the solid content of the cylinder, the hemisphere, and the cone, of equal base and altitude.**

APPLIC. *Let* BCAH, *be a square* BCA, *the fourth part of a circle, and* BCH, *a triangle,* (*fig.* 123,) *where* AC, = *the radius of the circular part* BH, *the base of the triangle, and the sides* CA, *and* CB, *are all equal; it is required to determine the ratio that the solids described by their revolution around the* BC, *as axis, will have to one another.*

Determination. The triangle CBH, by revolving upon BC, as axis, will describe a cone, the radius of the circular base of which will be BH, and the altitude BC, which will be equal to one another, thence produce an upright rectangular cone. The quadrant BAC, will describe a hemisphere, which will have the same radius as the base of the cone. The square $ACBH$, will describe a cylinder, having the same CA, for the radius of its base, and its equal CB, for its altitude. Any section like $DEFG$, being made through the three bodies perpendicular to their common axis, and another plane passing through $defg$, being parallel to this, the distance Dd, between them, being such a minute subdivision of the radius as has been used in the preceding cases; there will be such parts of the three solids generated as having the same altitude Dd, Ee, or Ff, and which on account of this supposed subdivision, will form cylinders of equal altitudes: therefore will be to one another as their

circular bases; that is: as the squares of their radii. The triangles CDE, and CDF, are right angled in D; they give, therefore, by § 36, of Plane Geometry, the following comparison between the squares of their sides, viz:

$$CF^2 = DC^2 + DF^2.$$

The triangles CBH, and CDE, are similar; thence $CD = DB$; the radii BH, BC, CF, and CA, are equal; DG, is parallel to CA, or BH, and equal to them, and to CF; the substitution of these equal quantities in the above, will give

$$DG^2 = DE^2 + DF^2.$$

Such is, therefore, the ratio of the similar surfaces described by these lines, therefore of the circles that are described with them as radii; and if multiplied all through by the same quantity, their ratio will be the same.

By the revolution of the triangle, the quadrant, and the square, these lines describing such circles, if the surfaces, the ratio of which is represented by these squares, be multiplied by a common factor, as $Dd = Ee = Ff = Gg$, which has been taken as a subdivision of the radius CB, such as to consider the bodies described as cylinders of the same altitude, these cylinders will have the same ratio as their bases; there will therefore be

$$DG^2.\ Ff = DE^2.\ Ff + DF^2.\ Ff,$$

And multiplying them as in the foregoing section, by the altitude divided by these Ff, considered as units, in which the radius CB, is expressed; this CB, being the common altitude of the three solids, and called $= h$, the result will be

$$\frac{BH^2.\ h.\ Ff}{Ff} = \frac{DG^2.\ h.\ Ff.}{Ff} = \frac{DE^2.\ h.\ Ff}{Ff} + \frac{DF^2.\ h.\ Ff}{Ff.}$$

or,
$$BH^2.\ h = DG^2.\ h = DE^2.\ h + DF^2.\ h.$$

The similarity of the triangles CDE, and CBH, gives the ratio of the whole the same as that of the parts. The BH^2, represents the base of the cylinder; the DE^2, that of the cone; and the DF^2, that of the hemisphere, described by the revolution of the square, triangle and quadrant, in any of their corresponding parts; thence the cylinder will be equal to the sum of the cone and hemisphere together.

It has been proved, § 27, that the cone was the third part of the cylinder, of equal base and altitude; therefore, the hemisphere will be equal to the other two thirds, which make the complement of the cone to the cylinder. If here, again, the circles forming the bases are expressed by the § 83, of Plane Geometry, and multiplied by the same radius, as altitude, the results of the above will be

$$\pi r3 = \frac{\pi r^3}{3} + \frac{2.\pi r^2.}{3}$$

That is, the cone is the one third part, and the hemisphere the two thirds of the cylinder, of equal base and altitude; or the ratio of the cone, hemisphere and cylinder, is 1, 2 and 3, which was to be determined.

Corol. What has been deduced here for the hemisphere, and the cone and cylinder corresponding to it, applies of course equally to the whole spheres as similar figures of equal altitudes, the multiple of which the double of that supposed before; thence the cone, sphere and cylinder, of equal altitude and upon the same great circle, are also in the ratio of 1, 2, and 3.

ERRATA.

Page.	*Line.*	*Error.*	*Correction.*
9,	32,	it guides,	they guide.
24,	7,	make their sum,	make angles, the sum of which is.
30,	2,	sides are,	side is.
41,	10,	triangle,	parallelogram.
—	25,	*EI*,	*E*.
47,	22,	*GFH*,	*GBH*.
56,	7,	*CI*,	*CE*.
59,	15,	than the,	than two.
60,	11,	*DCA*, there,	*DAC*, there.
71,	14,	in as many,	in twice as many.
72,	17,	*cD*,	*CD*.
85,	34,	the greater part of it with *CE*,	with *CE*, the greater part of it.
109,	26,	the two,	to the two.
110,	21,	qualities,	equalities.
124,	31,	angles,	angle.
131,	1,	set,	(strike out.)
132,	17,	to another,	to one another.
134,	5,	(add at end,)	as the pyramids are of the prisms of equal base and altitude.
145,	21,	supposition,	superposition.
150,	10,	the pyramids,	the similar pyramids.

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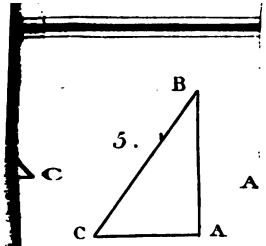
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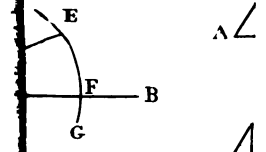
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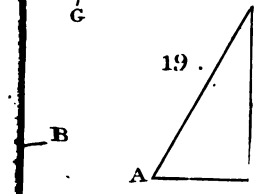
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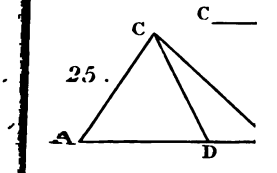
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4. Of the similarity and proportionality of solids, - - - - 135
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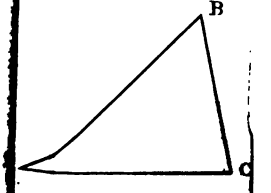

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5.
C
A
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A

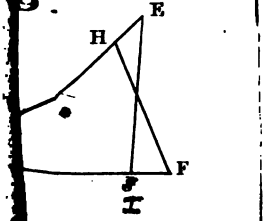

D

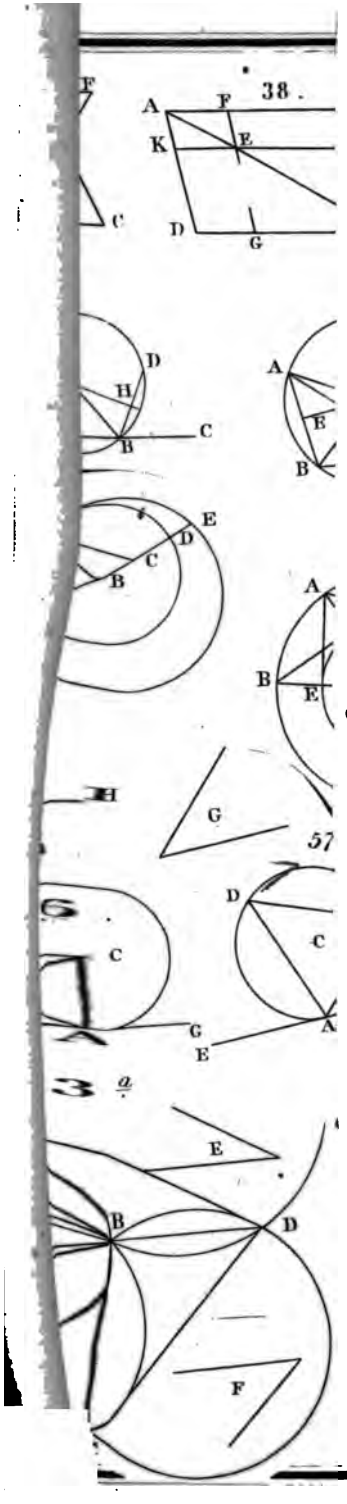

E
A
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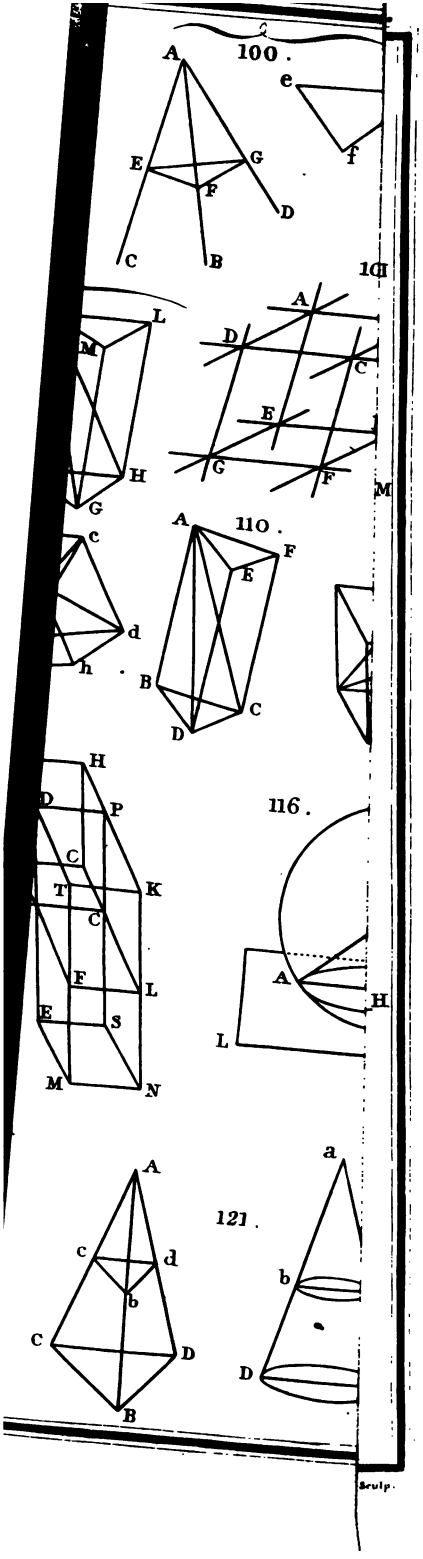

19.
B
A


C
C
25.
A
D


B
C


9.
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b
C
D
B
D



